Characterization of the stability range of the Hilbert transform with applications to spectral factorization

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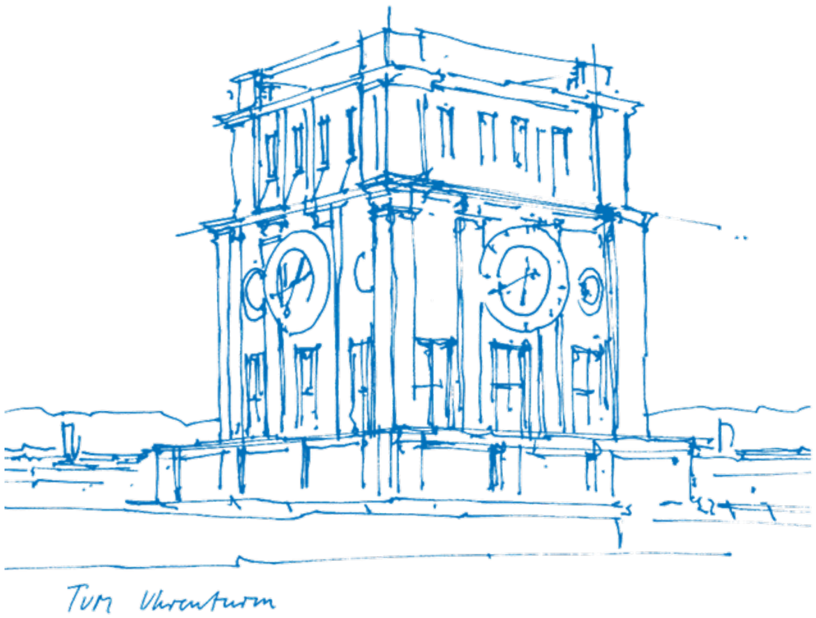

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Introduction

We consider the problem of calculating numerically the (finite) Hilbert transform

$$\widetilde{f}(t) = (\mathrm{H}f)(t) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\varepsilon \le |t-\tau| \le \pi} \frac{f(\tau)}{\tan([t-\tau]/2)} \mathrm{d}\tau \,, \qquad t \in [-\pi, \pi) \,. \tag{HT}$$

- This transformation plays an important role in science and engineering.
- In physics H is known as Kramers-Kronig relation.
- It is related to causality:
  - The real and imaginary part of a causal signal is related by the Hilbert transform.
  - The phase of a causal signal is determined by its amplitude.
  - Prediction and estimation of stationary time series – spectral factorization.

Challenges

- Singular integral kernel $\Rightarrow$ principal value integral in (HT)
- Calculation on digital computers $\Rightarrow$ calculation of (HT) has to be based on finitely many samples $\{f(\lambda_n)\}_{n=1}^{N}$ of the function f.

Hilbert Transform Approximations

Hilbert Transform: $$\widetilde{f}(t) = (\mathrm{H}f)(t) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\varepsilon \le |t-\tau| \le \pi} \frac{f(\tau)}{\tan([t-\tau]/2)} \mathrm{d}\tau\,, \qquad t \in [-\pi, \pi)\,. \qquad \text{(HT)}$$

- Given a sequence $\{\Lambda_N\}_{N\in\mathbb{N}}$ of sampling sets:

$$\Lambda_N = \{\lambda_1, \lambda_2, \ldots, \lambda_N\} \subset \mathbb{T} = [-\pi, \pi)\,, \qquad N \in \mathbb{N}\,.$$

- Design a sequence $\{\mathrm{H}_N\}_{N=1}^{\infty}$ of bounded linear operators H_N (each H_N is concentrated on Λ_N) such that

$$\lim_{N\to\infty} \left\| \mathrm{H}_N f - \mathrm{H} f \right\|_\infty = \lim_{N\to\infty} \max_{t\in[-\pi,\pi)} \left| (\mathrm{H}_N f)(t) - (\mathrm{H} f)(t) \right| = 0 \qquad \text{for all } f \in \mathscr{B}\,,$$

wherein $\mathscr{B}$ is our signal space (which has to be specified).

Question

For which signal spaces $\mathscr{B}$ there do exist such approximation sequences $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$?

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Example: Hilbert Transform on $L^2(\mathbb{T})$

- Let $f \in L^2(\mathbb{T})$ be a square integrable function on the *unit circle* $\mathbb{T} = [-\pi, \pi)$.
- f can be represented by its Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n(f)\, \mathrm{e}^{\mathrm{i}nt} \qquad \text{with Fourier coefficients} \qquad c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\tau)\, \mathrm{e}^{-\mathrm{i}n\tau}\, \mathrm{d}\tau$$

- Its harmonic *conjugate* $\widetilde{f}$ is given by the Hilbert transform of f

$$\widetilde{f}(t) = (\mathrm{H}f)(t) = -\mathrm{i} \sum_{n=-\infty}^{\infty} \mathrm{sgn}(n)\, c_n(f)\, \mathrm{e}^{\mathrm{i}nt} \qquad \text{with} \qquad \mathrm{sgn}(n) = \begin{cases} -1 & : \ n < 0 \\ 0 & : \ n = 0 \\ 1 & : \ n > 0 \end{cases}$$

Properties

- Hilbert transform is bounded mapping $\mathrm{H} : L^p(\mathbb{T}) \to L^p(\mathbb{T})$, $1 < p < \infty$.
- The Hilbert transform is a bounded mapping $\mathrm{H} : L^\infty(\mathbb{T}) \to BMO$.
- For $f \in \mathscr{C}(\mathbb{T})$, we have $\widetilde{f} = \mathrm{H}f \in L^p(\mathbb{T})$ for every $1 \leq p < \infty$ but $\widetilde{f} = \mathrm{H}f \notin \mathscr{C}(\mathbb{T})$, in general.

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Example of a Hilbert Transform Approximation

- For every $N \in \mathbb{N}$, we consider the equidistant sampling set

$$\Lambda_N = \left\{\lambda_{N,k} = k\tfrac{\pi}{N} : k = 0, 1, \dots, 2N-1\right\}$$

- First, we approximate $f \in L^2(\mathbb{T})$ by its partial Fourier series

$$(\mathrm{D}_N f)(t) = \sum_{n=-N+1}^{N-1} c_{N,n}(f)\,\mathrm{e}^{\mathrm{i}nt}$$

but where we exchanged the exact Fourier coefficients $c_n(f)$ for approximations $c_{N,n}(f)$.

- The approximations $c_{N,n}(f)$ are obtained by replacing the integral in the formula for the Fourier coefficients with the left Riemann sum with nodes Λ_N.

$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\,\mathrm{e}^{-\mathrm{i}n\tau}\,\mathrm{d}\tau \qquad \mapsto \qquad c_{N,n}(f) = \tfrac{1}{2N}\sum_{k=0}^{2N-1} f(\lambda_{N,k})\,\mathrm{e}^{-\mathrm{i}\pi nk/N}$$

- To get an approximation of $\widetilde{f} = \mathrm{H}f$, we apply H to the trigonometric polynomial $\mathrm{D}_N f$

$$(\widetilde{\mathrm{D}}_N f)(t) := (\mathrm{H}\mathrm{D}_N f)(t) = -\mathrm{i}\sum_{n=-(N-1)}^{N-1} \operatorname{sgn}(n)\,c_{N,n}(f)\,\mathrm{e}^{\mathrm{i}nt} = \sum_{k=0}^{2N-1} f(\lambda_{N,k})\,\widetilde{\mathscr{D}}_N\!\left(t - k\frac{\pi}{N}\right).$$

with the kernel $\widetilde{\mathscr{D}}_N(t) = \frac{1}{N}\sum_{n=1}^{N-1}\sin(nt)$.

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Problem Statement

The above defined sequence $\{\widetilde{\mathrm{D}}_N\}_{N\in\mathbb{N}}$ satisfies

$$\lim_{N\to\infty} \left\|\widetilde{\mathrm{D}}_N f - \mathrm{H}f\right\|_{L^2(\mathbb{T})} = 0 \qquad \text{for all} \qquad f \in L^2(\mathbb{T}) .$$

Questions

- For which subset $\mathscr{B} \in L^2(\mathbb{T})$ do we even have

$$\lim_{N\to\infty} \left\|\widetilde{\mathrm{D}}_N f - \mathrm{H}f\right\|_{\infty} = 0 \qquad \text{for all} \qquad f \in \mathscr{B} .$$

- More general: For which spaces $\mathscr{B} \subset L^2(\mathbb{T})$ is it possible to find sequences of bounded linear operators $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ such that

$$\lim_{N\to\infty} \left\|\mathrm{H}_N f - \mathrm{H}f\right\|_{\infty} = 0 \qquad \text{for all} \qquad f \in \mathscr{B} \,?$$

- Which properties of $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ are necessary/sufficient for convergence on $\mathscr{B}$?

Uniform Norm

- to control peak value of the approximation: hardware requirements (dynamic range)
- relevant for continuous functions
- stability norm $L^2(\mathbb{T}) \to L^2(\mathbb{T})$

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Outline of the Paper

1. We introduce a scale of Banach space $\{\mathscr{B}_\beta\}_{\beta\geq 0}$ of continuous functions of finite energy.
   - These are „good“ for the Hilbert transform.
   - The parameter $\beta \geq 0$ characterizes the energy concentration of the signals.
2. We introduce a class of sampling based Hilbert transform approximations $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$.
   - This class is characterizes by three simple axioms.
   - This class contains basically all practically relevant Hilbert transform approximation methods.
3. Divergence results for the spaces $\mathscr{B}_\beta$ with $\beta \leq 1$.
   - For these spaces, there exists no Hilbert transform approximation in our class.
4. Convergence results for spaces $\mathscr{B}_\beta$ with $\beta > 1$.
   - For these spaces, there always exist a Hilbert transform approximation in our class.
   - Simple examples of convergent methods can be found.

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Signal Spaces

Space of all continuous functions $f \in \mathscr{C}(\mathbb{T})$ with a continuous conjugate $\widetilde{f}$

$$\mathscr{B} := \left\{ f \in \mathscr{C}(\mathbb{T}) \;:\; \widetilde{f} = \mathrm{H}f \in \mathscr{C}(\mathbb{T}) \right\} \qquad \text{with norm} \qquad \|f\|_{\mathscr{B}} = \max\left(\|f\|_\infty, \|\mathrm{H}f\|_\infty\right)$$

- The Hilbert transform $H : \mathscr{B} \to \mathscr{B}$ is well defined and bounded.

$L^2(\mathbb{T})$ subpaces with energy concentration

Any $f \in L^2(\mathbb{T})$ can be represented by the trigonometric series

$$f(t) = \frac{a_0(f)}{2} + \sum_{n=1}^{\infty} a_n(f)\cos(nt) + b_n(f)\sin(nt) \qquad \text{with} \qquad \begin{aligned} a_n(f) &= \textstyle\int_{\mathbb{T}} f(\tau)\cos(n\tau)\,\mathrm{d}\tau \\ b_n(f) &= \textstyle\int_{\mathbb{T}} f(\tau)\sin(n\tau)\,\mathrm{d}\tau \end{aligned}$$

For $\beta \geq 0$, we define

$$\mathscr{L}_\beta := \left\{ f \in L^2(\mathbb{T}) \;:\; \sum_{n\in\mathbb{Z}} n(\log n)^\beta \left[|a_n(f)|^2 + |b_n(f)|^2\right] < \infty \right\}$$

- $\beta \geq 0$ characterizes the smoothness of the functions $f \in \mathscr{L}_\beta$: As larger β as smoother f.
- $\mathscr{L}_0$ corresponds to Sobolev space $\mathscr{H}^{1/2} = \mathscr{W}^{1/2,2}$.
- For $\beta > 1$, one has $\mathscr{L}_\beta \subset \mathscr{C}(\mathbb{T})$
- $\beta \geq 0$ characterizes the energy concentration. As larger β as better the concentration.

Our Scale of Signal Spaces

Our signal spaces are defined for all $\beta \geq 0$ as the intersection of the previous two spaces

$$\mathscr{B}_\beta := \mathscr{L}_\beta \cap \mathscr{B} \qquad \text{with norm} \qquad \|f\|_\beta = \|f\|_{\mathscr{B}} + \left(\sum_{n \in \mathbb{Z}} n (\log n)^\beta \left[|a_n(f)|^2 + |b_n(f)|^2 \right] \right)^{1/2}.$$

- Each $\mathscr{B}_\beta$ is a Banach space
- Every $f \in \mathscr{B}_\beta$ is continuous with a continuous conjugate
- Every $f \in \mathscr{B}_\beta$ has finite $L^2(\mathbb{T})$ - energy
- Every $f \in \mathscr{B}_\beta$ has finite Dirichlet energy
- $\mathscr{B}_{\beta_2} \subset \mathscr{B}_{\beta_1} \subset \mathscr{B}_0 = \mathscr{H}^{1/2} \subset \mathscr{B} \subset \mathscr{C}(\mathbb{T})$ for all $\beta_2 > \beta_1 > 0$.
- The parameter $\beta \geq 0$ characterizes the energy concentration.

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Relation to the Dirichlet Problem

Dirichlet Problem on the Unit Circle

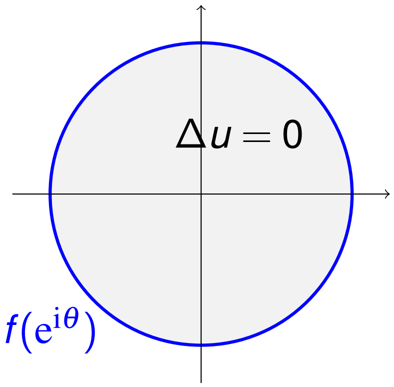
Let f be a given function on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We look for an u inside the unit circle $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that

1. $\frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z) = (\Delta u)(z) = 0$ for all $z = x + \mathrm{i}y \in \mathbb{D}$
2. $u(\mathrm{e}^{\mathrm{i}t}) = f(\mathrm{e}^{\mathrm{i}t})$ for all $t \in \mathbb{T} = [-\pi, \pi)$

Dirichlet's Principle

The solution of the Dirichlet problem can be obtained by minimizing the Dirichlet energy

$$D(u) = \frac{1}{2\pi} \iint_{\mathbb{D}} \left\| (\operatorname{grad} u)(z) \right\|_{\mathbb{R}^2}^2 \mathrm{d}z = \sum_{n=-\infty}^{\infty} |n| \, |c_n(f)|^2 = \|f\|_{\mathscr{H}^{1/2}}^2$$



- The boundary function of solutions of the Dirichlet problem belongs to the Sobolev space $\mathscr{H}^{1/2}$.
- If f is additionally in $\mathscr{B}$ then $f \in \mathscr{B}_0$.

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A Class of Hilbert Transform Approximations

We consider sequences $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ of bounded linear operators $\mathrm{H}_N : \mathscr{B} \to \mathscr{B}$ which satisfy the following three axioms:

(A) Concentration on a sampling set:

To every $N \in \mathbb{N}$ there exists a finite set $\Lambda_N = \{\lambda_{N,n} : n = 1,\dots,M_N\} \subset \mathbb{T}$ such that for all $f_1, f_2 \in \mathscr{B}$

$$f_1(\lambda_{N,n}) = f_2(\lambda_{N,n}) \qquad \text{for all } \lambda_{N,n} \in \Lambda_N$$
$$\text{implies } (\mathrm{H}_N f_1)(t) = (\mathrm{H}_N f_2)(t) \qquad \text{for all } t \in \mathbb{T}\,.$$

(B) Weak convergence on $\mathscr{B}$:

For every $f \in \mathscr{B}$, the sequence $\{\mathrm{H}_N f\}_{N\in\mathbb{N}}$ converges weakly to $\mathrm{H}f$, i.e.

$$\lim_{N\to\infty} \langle \mathrm{H}_N f, \varphi \rangle_2 = \langle \mathrm{H} f, \varphi \rangle_2 \qquad \text{for all } \varphi \in \mathscr{C}^\infty(\mathbb{T})\,.$$

(C) Zero mapping for constant functions:

$\mathrm{H}_N 1 = 0$ for all $N \in \mathbb{N}$, with the constant function $1(t) = 1$ for all $t \in \mathbb{T}$.

Remark:

If $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ satisfies Axiom (A) then each H_N has the form

$$(\mathrm{H}_N f)(t) = \sum_{n=1}^{M_N} f(\lambda_{N,n})\, h_{N,n}(t) \qquad \text{with} \qquad \{h_{N,1}, h_{N,2}, \dots, h_{N,M_N}\} \subset \mathscr{B}\,.$$

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A Strong Divergence Results

Theorem

Let $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be a sequence of bounded linear operators $\mathrm{H}_N : \mathscr{B} \to \mathscr{B}$ which satisfies Axioms (A), (B), and (C). Then for any $0 \le \beta \le 1$ there exists an $f_* \in \mathscr{B}_\beta$ and a sequence $\{\theta_N\}_{N\in\mathbb{N}} \subset \mathbb{T}$ such that

$$\lim_{N\to\infty} \left\| \mathrm{H}_N \mathrm{T}_{\theta_N} f_* \right\|_\infty = +\infty \,.$$

with the translation operator $\mathrm{T}_\theta : \mathscr{B} \to \mathscr{B}$ given by $(\mathrm{T}_\theta f)(t) = f(t-\theta)$.

Remarks

- The numerical calculation is unstable with respect to jitter:
  If for $f \in \mathscr{B}$ only $f_\varepsilon = \mathrm{T}_\varepsilon f$ is known, then $\|\mathrm{H}_N f_\varepsilon - \mathrm{H} f\|_\infty$ may get arbitrarily large even for large N.
- Strong divergence: There exists no convergent subsequence.

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Weak Divergence on all Spaces $\mathscr{B}_\beta$ with $\beta \in [0,1]$

Corollary

Let $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ be an arbitrary sequence of bounded linear operators $\mathrm{H}_N : \mathscr{B} \to \mathscr{B}$ which satisfies Axioms (A) – (C). Then for each $0 \le \beta \le 1$ holds: To every sequence $\{N_k\}_{k\in\mathbb{N}}$ there exists an $f_* \in \mathscr{B}_\beta$ such that

$$\limsup_{k\to\infty} \left\| \mathrm{H}_{N_k} f_* \right\|_\infty = +\infty \, .$$

Remarks

- This result implies in particular that in every space $\mathscr{B}_\beta$ with $\beta \in [0,1]$ there exists a function f_* with

$$\limsup_{N\to\infty} \left\| \mathrm{H}_N f_* \right\|_\infty = +\infty \qquad \text{and} \qquad \limsup_{N\to\infty} \left\| \mathrm{H}_N f_* - \mathrm{H} f_* \right\|_\infty = +\infty \, .$$

- There is no sampling based Hilbert transform approximation on the spaces $\mathscr{B}_\beta$ with $0 \le \beta \le 1$.
- In particular, not on the set of all solutions of the Dirichlet problem (finite Dirichlet energy).
- We only have weak divergence,
  i.e. to every $f \in \mathscr{B}_\beta$ there may exist a subsequence $\{N_k = N_k(f)\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \left\| \mathrm{H}_{N_k} f \right\|_\infty < \infty \qquad \text{or even} \qquad \lim_{k\to\infty} \left\| \mathrm{H}_{N_k} f - \mathrm{H} f \right\|_\infty = 0 \, .$$

Weak Divergence versus Strong Divergence

- Given an approximation sequence $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ which diverges weakly, i.e.

$$\limsup_{N\to\infty} \left\|\mathrm{H}_N f - \mathrm{H} f\right\|_\infty = \infty \qquad \text{for some } f \in \mathscr{B}_\beta \,.$$

  Then there may exist a subsequence $\{N_k = N_k(f)\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \left\|\mathrm{H}_{N_k} f - \mathrm{H} f\right\|_\infty = 0 \,.$$

  Then $\{\mathrm{H}_{N_k(f)}\}_{k\in\mathbb{N}}$ is a convergent approximation method adapted to f.
- Assume $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ diverges strongly

$$\lim_{N\to\infty} \left\|\mathrm{H}_N f - \mathrm{H} f\right\|_\infty = \infty \qquad \text{for some } f \in \mathscr{B}_\beta \,.$$

  Then no convergent subsequence exists $\Longrightarrow$ adaption does not help.

- ... every sequence $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ which diverges weakly on $\mathscr{B}_\beta$ $\Rightarrow$ there exists no non-adaptive approximation methods on $\mathscr{B}_\beta$
- ... every sequence $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ which diverges strongly on $\mathscr{B}_\beta$ $\Rightarrow$ there exists no adaptive (and non-adaptive) approximation methods on $\mathscr{B}_\beta$

Spaces with Convergent Approximation Methods

Theorem

For any $\beta > 1$ there exit sequences $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ of bounded linear operators $\mathrm{H}_N : \mathscr{B} \to \mathscr{B}$ which satisfy Axioms (A)–(C) such that

$$\lim_{N\to\infty} \left\|\mathrm{H}_N f - \mathrm{H} f\right\|_\infty = 0 \qquad \text{for all } f \in \mathscr{B}_\beta \,.$$

- If the energy of the signals is sufficiently concentrated then there always exist sampling based approximation methods which converges for all signals in the space $\mathscr{B}_\beta$ with $\beta > 1$.
- Theorem can be proved by a constructing particular method.

Characterization of Convergent Method

Theorem

Let $\beta > 1$ *and let* $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ *be a sequence of bounded linear operators* $\mathrm{H}_N : \mathscr{B} \to \mathscr{B}$ *such that*

1. *For every* $n \in \mathbb{N}$ *holds*

$$\lim_{N\to\infty} \left\| \mathrm{H}_N[\cos(n\cdot)] - \sin(n\cdot) \right\|_\infty = 0 \qquad \text{and} \qquad \lim_{N\to\infty} \left\| \mathrm{H}_N[\sin(n\cdot)] + \cos(n\cdot) \right\|_\infty = 0 \,.$$

2. *There exists a constant* C *such that*

$$\max\left(\left\| \mathrm{H}_N[\cos(n\cdot)] \right\|_\infty, \left\| \mathrm{H}_N[\sin(n\cdot)] \right\|_\infty \right) \leq C \qquad \text{for all } N \in \mathbb{N} \,.$$

Then one has

$$\lim_{N\to\infty} \left\| \mathrm{H}_N f - \mathrm{H} f \right\|_\infty = 0 \qquad \text{for all } f \in \mathscr{B}_\beta \,.$$

Thus, if an approximation method $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$

- converges for the sine- and cosine functions (i.e. for the pure frequencies), and
- if the approximations of the pure frequencies are uniformly bounded

then the method $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ converges for all $f \in \mathscr{B}_\beta$ with $\beta > 1$.

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A Convergent Hilbert Transform Approximation

We consider again the sequence $\{\widetilde{\mathrm{D}}_N\}_{N\in\mathbb{N}}$ of the sampled conjugate Fourier series from the beginning

$$(\widetilde{\mathrm{D}}_N f)(t) := (\mathrm{H}\mathrm{D}_N f)(t) = -\mathrm{i}\sum_{n=-(N-1)}^{N-1} \operatorname{sgn}(n)\, c_{N,n}(f)\, \mathrm{e}^{\mathrm{i}nt} = \sum_{k=0}^{2N-1} f(\lambda_{N,k})\, \widetilde{\mathscr{D}}_N\Big(t - k\frac{\pi}{N}\Big)\,,$$

with the conjugate Dirichlet kernel $\widetilde{\mathscr{D}}_N(t) = \frac{1}{N}\sum_{n=1}^{N-1}\sin(nt)$ and which are concentrated on the equidistant sampling sets

$$\Lambda_N = \Big\{\lambda_{N,k} = k\tfrac{\pi}{N} \,:\, k = 0,1,\dots,2N-1\Big\}\,.$$

It is fairly easy to show that this sequence $\{\widetilde{\mathrm{D}}_N\}_{N\in\mathbb{N}}$

- satisfies Axioms (A), (B) and (C)
- has the two properties of the previous theorem which characterized all convergent methods

and so, we have

$$\lim_{N\to\infty} \big\|\widetilde{\mathrm{D}}_N f - \mathrm{H}f\big\|_\infty = 0 \qquad \text{for all } f\in\mathscr{B}_\beta \quad \text{with} \quad \beta > 1\,.$$

Conclusions

- We introduced a scale of Banach spaces $\mathscr{B}_\beta$, $\beta \geq 0$ of functions
  - which are continuous with a continuous conjugate
  - with finite (Dirichlet) energy
  - with different energy concentration, characterized by β

- In the scale $\{\mathscr{B}_\beta\}_{\beta \geq 0}$, we characterized precisely those spaces on which
  - there do not exist any sampling based linear Hilbert transform approximations: $\beta \in [0, 1]$
  - there do exist sampling based Hilbert transform approximations: $\beta > 1$.

- For $\beta > 1$ even very simple approximations methods (sampled conjugate Fourier series) work

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Thank You! – Questions?