Complexity Blowup if Continuous-Time LTI Systems are Implemented on Digital Hardware

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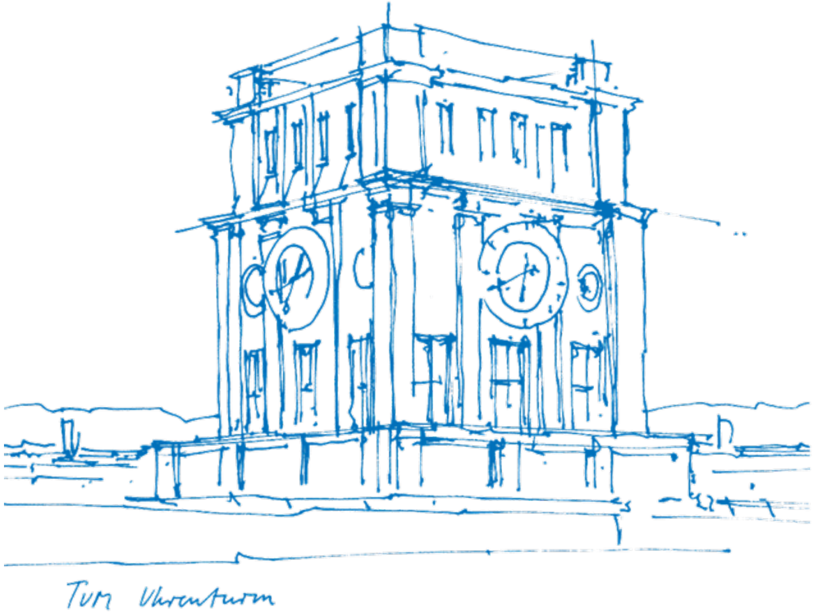


Outline of the Talk

What is the computational complexity of simulating causal, time-invariant linear systems on digital computers?

Having low-complexity input signal. What is the complexity for calculating the output signal?

Outline

1. Linear Time-Invariant (LTI) Systems
2. Computability and Complexity
3. Examples of Complexity Blowup
4. Summary

Linear Time-Invariant (LTI) Systems

First-Order Linear Time-Invariant (LTI) Systems

- We consider causal, linear, time-invariant systems of first order with input $x(t)$ and output $y(t)$.
- Input-Output relation is described by linear differential equation with constant coefficients

$$\begin{aligned} & y'(t) + \alpha_0\, y(t) = \beta_1 x'(t) + \beta_0 x(t)\,, \quad t > 0 \\ & \text{with initial condition} \quad y(0) = y_0 \quad \text{and} \quad x(0) = x_0\,, \end{aligned} \tag{1}$$

  with coefficients $\alpha_0, \beta_0, \beta_1, x_0, y_0 \in \mathbb{R}$, which are assumed to be polynomial-time computable.
- The unique solution of (1) is given for $t > 0$ by the closed form expression

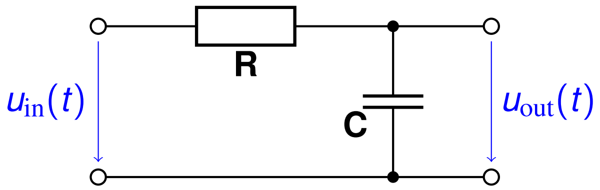
$$y(t) = (\mathrm{S}x)(t) = y_0\, \mathrm{e}^{-\alpha_0 t} + a\left[x(t) - x_0\, \mathrm{e}^{-\alpha_0 t}\right] + b \int_0^t x(\tau)\, \mathrm{e}^{-\alpha_0(t-\tau)}\, \mathrm{d}\tau \tag{2}$$

  with the constants $a := \beta_1$ and $b := \beta_0 - \beta_1 \alpha_0$.

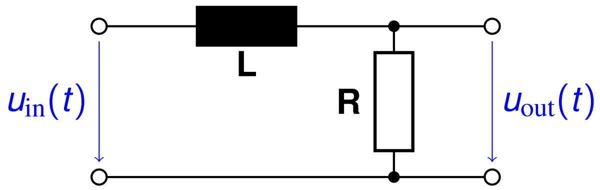
- Goal: Simulate the output signal $y(t)$, for $t \in [0,1]$ on a *digital computer*, for feasible (continuously differentiable, computable) signals $x(t)$.
- Question: Assume low-complexity input $x(t)$. What is the complexity for computing $y(t)$?

Examples of First Order LTI systems

Two examples for a general LTI system $\mathbb{S}$ with input $x(t) = u_{\mathrm{in}}(t)$ and output $y(t) = u_{\mathrm{out}}(t)$.



RC low pass with resistor R, capacitor C, and cutoff frequency $\omega_0 = (RC)^{-1}$



LR low pass with inductor L, resistor R, and cutoff frequency $\omega_0 = R/L$

Computability

Computable Rational Numbers

Definition: A sequence $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{Q}$ of rational numbers is said to be computable if there exist recursive functions $a, b, s : \mathbb{N} \to \mathbb{N}$ with $b(n) \neq 0$ and such that

$$r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}\,, \qquad n \in \mathbb{N}\,.$$

A recursive function $a : \mathbb{N} \to \mathbb{N}$ is a mapping that is build form elementary computable functions and recursion and can be calculated on a *Turing machine*.

Turing machine

- can simulate any given algorithm and therewith provide a simple but very powerful model of computation.
- is a theoretical model describing the fundamental limits of any realizable digital computer.
- Most powerful programming languages are called Turing-complete (such as C, C++, Java, etc.).

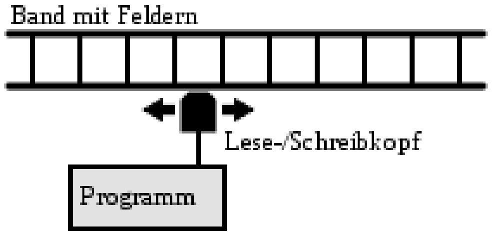


Figure taken from *Wikipedia*

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proc. London Math. Soc.*, vol. s2-42, no. 1, 1937.

Computable Real Numbers

- ▷ Any real number $t \in \mathbb{R}$ is the limit of a sequence of rational numbers.
- ▷ For $t \in \mathbb{R}$ to be computable, the convergence has to be effective.

Definition (Computable number): A number $t \in \mathbb{R}$ is said to be *computable* if there exist a recursive function $\gamma : \mathbb{N} \to \mathbb{Q}$ such that

$$|t - \gamma(n)| \leq 2^{-n}, \qquad \text{for all } n \in \mathbb{N}.$$

In this case, we say that γ *binary converges* to t.

$\Rightarrow$ $x \in \mathbb{R}$ is computable if a Turing machine can approximate it with exponentially vanishing error.

- $\mathbb{R}_c$ stand for the set of all *computable real numbers*.
- Note that the set of computable numbers $\mathbb{R}_c \subsetneq \mathbb{R}$ is only countable.

Computable Functions

- We consider function-oracle Turing machines: Ordinary Turing machine TM with an additional *function-oracle* γ
- The function oracle is able to calculate the function value γ in a single operation.
- We neglect the computational complexity for determine t in calculating $x(t)$.

Definition: A function $x : [a,b] \to \mathbb{R}$ is said to be *computable* on the interval $[a,b] \subseteq \mathbb{R}$ if there exists a function-oracle Turing machine TM so that for each $t \in [a,b]$ and each γ that binary converges to t, the function $\widetilde{x}(n) = \mathrm{TM}_\gamma(n)$ computed by TM with oracle γ binary converges to $x(t)$, i.e. if

$$\left| x(t) - \mathrm{TM}_\gamma(n) \right| \leq 2^{-n}, \qquad \text{for all } n \in \mathbb{N}\,.$$

Remark:
If $x : [a,b] \to \mathbb{R}$ is a computable function on $[a,b]$ then $x \in \mathscr{C}([a,b])$, i.e. x is a continuous function on $[a,b]$.

Computational Complexity

Definition: Let $x : [a,b] \to \mathbb{R}$ be a computable function. We say that the complexity of x is bounded by a function $q : \mathbb{N} \to \mathbb{N}$ if there exists a function–oracle Turing machine TM_γ, which computes x so that for all γ that binary converge to $t \in [a,b]$ and for all $n \in \mathbb{N}$, $\mathrm{TM}_\gamma(n)$ satisfies

$$\left| x(t) - \mathrm{TM}_\gamma(n) \right| < 2^{-n}$$

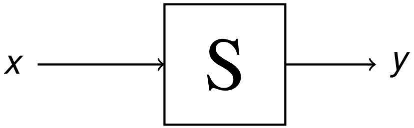
after a computation time of at most $q(n)$.
The function $x : [a,b] \to \mathbb{R}$ is said to be polynomial-time computable if its complexity is bounded by a polynomial q.

Complexity Classes

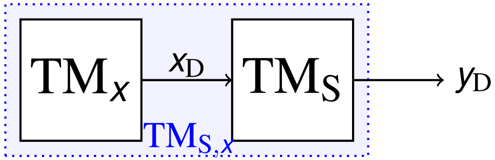
- *FP* The class of functions, which can be computed by an function-oracle Turing machine in polynomial time.
- $\#P$ The class of functions that enumerate the number of accepting computations of polynomial-time function-oracle Turing machines.

Assumption: $FP \subsetneq \#P$ – Note that $FP = \#P$ would imply $P = NP$.

Simulation of LTI Systems on Digital Computers



A general LTI system S with input signal x and output signal y.



A Turing machine implementation $\mathrm{TM}_{\mathrm{S},x}$ for simulating S for input x consisting of a signal generator TM_x and a Turing machine TM_{S} which simulates the behavior of S.

- ▷ Digital computers can calculate exactly only with rational numbers
- ▷ Signal generator TM_x prepares the input signal x up to n significant digits

$$|x(t) - x_D(t)| < 2^{-n}$$

- ▷ Assume low complexity input signal, i.e generation of x_D needs time $p_x(n)$ with a certain polynomial p_x.
- ▷ Question: Is the output y_{D} again a low complexity signal?
  Can $y_D(t)$ with $|(\mathrm{S}x)(t) - y_D(t)| < 2^{-m}$ be computed in time $p_y(m)$ with a certain polynomial p_y?

Complexity Blowup

Complexity Blowup

We consider causal continuous-time linear systems S mapping functions on $\mathbb{R}_+$ onto functions on $\mathbb{R}_+$.

$$(\mathrm{S}x)(t) = ax(t) + b\int_0^t x(\tau)\,h(t-\tau)\,\mathrm{d}\tau\,, \quad t > 0\,. \tag{3}$$

Definition: Let $\mathrm{S} : L^\infty([0,1]) \to L^\infty([0,1])$ be an LTI system with input-output relation (3). We say that S shows complexity blowup, if there exists a polynomial-time computable input signal $x_* : [0,1] \to \mathbb{R}$ with the property that the corresponding output signal $y_* = \mathrm{S}x_*$ is not polynomial-time computable.
In this case, we say that S shows complexity blowup for the input signal x_*.

Complexity Blowup – Results

We consider causal continuous-time linear systems S mapping functions on $\mathbb{R}_+$ onto functions on $\mathbb{R}_+$.

$$(\mathrm{S}x)(t) = ax(t) + b\int_0^t x(\tau)\,h(t-\tau)\,\mathrm{d}\tau\,, \quad t > 0\,.$$

Theorem: Assume $FP \neq \#P$ and let S be a first-order LTI system described by (3) with polynomial-time computable coefficients. If $b = \beta_0 - \beta_1\alpha_0 \neq 0$ then S shows complexity blowup, i.e. there exists a polynomial-time computable signal $x_* : [0,1] \to \mathbb{R}$ so that the output $y_* = \mathrm{S}x_*$ is not polynomial-time computable on $[0,1]$.

Corollary: A first-order LTI system (1) with polynomial-time computable coefficients shows no complexity blowup if and only if $b = \beta_0 - \beta_1\alpha_0 = 0$.

- Complete characterization of all first-order systems showing complexity blowup.
- Only trivial cases ($b = 0$) show no complexity blowup
- Extension to higher-order systems and non-causal systems possible
- Proof is based on a result of *Friedman* and *Ko*.

Discussion – Practical Computability

- ▷ Applied science, applications, cryptogrophy $\Rightarrow$ (practically) "computability"versus "non-computable"
- ▷ (practically) "non-computable" if the calculation time, using the best possible algorithm, grows faster than any polynomial in the number of the parameters
- ▷ (practically) "computable" if the calculation time grows at post polynomially in the number of the parameters

Our results: For any first order, causal LTI system S (with $b \neq 0$) there exit input signals x_* which are practically computable but such that the output $y_* = \mathrm{S}x_*$ is practically non-computable.

Further Examples of Complexity Blowup

Calculation of Fourier Series Approximation

- $\mathscr{C}_{2\pi}$: be the set of all computable, continuous, and 2π-periodic functions
- $\mathscr{C}_{2\pi}^{\mathrm{pol}}$: be the set of all $x \in \mathscr{C}_{2\pi}$ which are polynomial-time computable
- For $x \in \mathscr{C}_{2\pi}$, we consider the problem of calculating for $N \in \mathbb{N}$ the partial Fourier series

$$(\mathrm{F}_N x)(t) = \frac{a_0}{2} + \sum_{k=1}^{N} [a_k \cos(kt) + b_k \sin(kt)]$$

with the *Fourier coefficients* of x given by

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x(\tau)\cos(k\tau)\,\mathrm{d}\tau \qquad \text{and} \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x(\tau)\sin(k\tau)\,\mathrm{d}\tau$$

Theorem: If $FP_1 \neq \#P_1$ then there exists an $x_* \in \mathscr{C}_{2\pi}^{\mathrm{pol}}$ such that there is an $N \in \mathbb{N}$ so that $(\mathrm{F}_N x_*)(t)$ is not polynomial-time computable for $t \in [-\pi, \pi)$.

Calculation of the Fourier Transform

- $\mathscr{C}(0,\pi)$: set of all computable continuous functions on $[0,\pi] \subset \mathbb{R}$
- $\mathscr{C}^{\text{pol}}(0,\pi)$: the set of all $x \in \mathscr{C}(0,\pi)$, which are polynomial-time computable
- For $x \in \mathscr{C}(0,\pi)$, we consider the problem of calculating the Fourier transform

$$\widehat{x}(\omega) = (\mathscr{F}x)(\omega) = \int_0^{\pi} x(\tau)\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\tau\,, \qquad \omega \in \mathbb{R}\,.$$

pointwise for some $\omega \in \mathbb{R}$.

Theorem: For any $\omega \in \mathbb{R}$ there exists an $x_* \in \mathscr{C}^{\text{pol}}(0,\pi)$ such that if $FP_1 \neq \#P_1$ then $\widehat{x}_*(\omega)$ is not polynomial-time computable.

Summary

Summary

- ▷ For every causal LTI system $\mathbb{S}$ there exist low complexity input signals x so that the corresponding output $y = \mathbb{S}x$ is high complexity.
- ▷ Other examples of complexity blowup: Calculation of Fourier series approximation and Fourier transform
- ▷ Extension to non-causal and higher-order LTI systems possible
- ▷ Notion of "practical non-computability": foundation of modern cryptography
  - polynomial-time computable $\Rightarrow$ is practically computable
  - not polynomial-time computable $\Rightarrow$ is practically non-computable
- ▷ For any (first-order) LTI system $\mathbb{S}$ (with $b \neq 0$) there exist practically computable input signals x_* so that the corresponding output signal $y_* = \mathbb{S}x_*$ is practically non-computable

H. Boche and V. Pohl, "Complexity Blowup in Simulating Analog Linear Time-Invariant Systems on Digital Computer," *IEEE Trans. Signal Processing*, vol. 69 (Aug. 2021), 5005–5020.