Common Randomness Generation from Finite Compound Sources

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Outline

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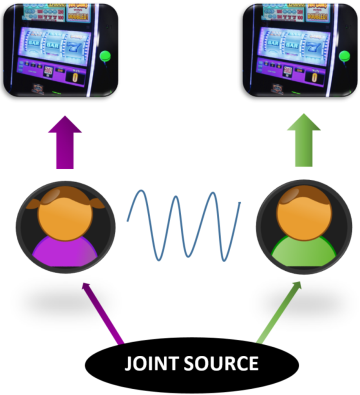
1. Overview
2. System Model for CR Generation from Finite Compound Sources
3. Single-Letter Bounds on the Compound CR Capacity
4. Proof Sketch of the Bounds
5. Conclusions

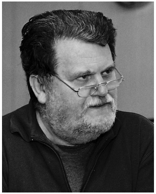
What is Common Randomness (CR)?



- The parties aim to agree on a common random variable with high probability

CR Generation

- Introduced by Ahlswede and Csiszár[1]



[1] R. Ahlswede and I. Csiszár, "Common randomness in information theory and cryptography. II. CR capacity," *IEEE Transactions on Information Theory*, vol. 44, no. 1, pp. 225-240, 1998.

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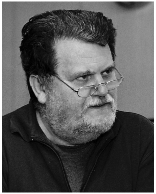
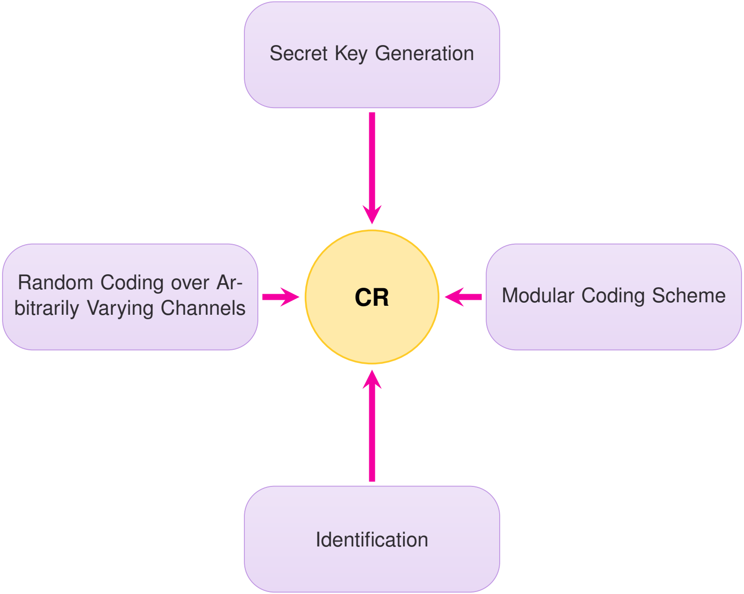


- Ahlswede and Csiszár established the CR capacity of rate-limited perfect channels as well as discrete memroyless channels[1]

[1] R. Ahlswede and I. Csiszár, "Common randomness in information theory and cryptography. II. CR capacity," *IEEE Transactions on Information Theory*, vol. 44, no. 1, pp. 225-240, 1998.

Applications of CR Generation



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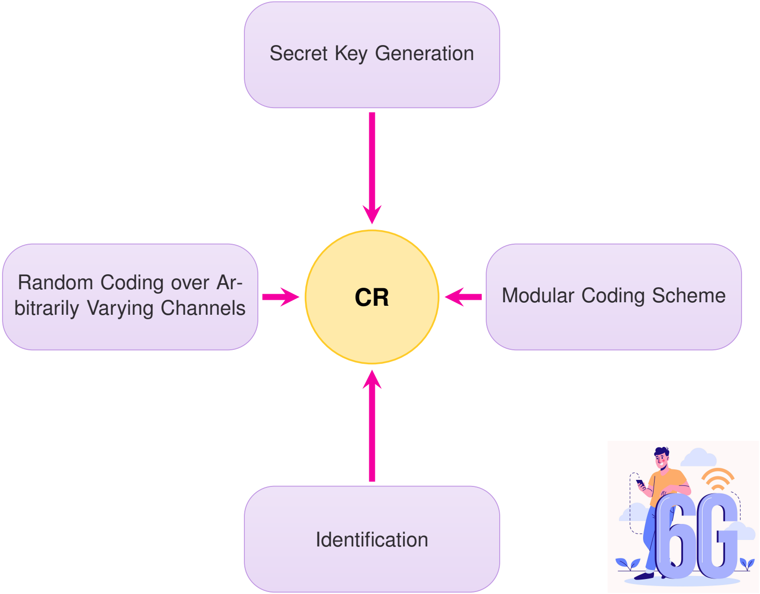

Applications of CR Generation

Secret Key Generation

Random Coding over Arbitrarily Varying Channels

CR

Modular Coding Scheme

Identification

CR Generation with One-Way Communication

Source Type	Channel Type	Known Result
finite	Gaussian	Single-letter CR capacity Formula[1]

[1] R. Ezzine, W. Labidi, H. Boche and C. Deppe, "Common Randomness Generation and Identification over Gaussian Channels," GLOBECOM 2020 - 2020 IEEE Global Communications Conference, Taipei, Taiwan, 2020.

[2] R. Ezzine, M. Wiese, C. Deppe and H. Boche, "Common Randomness Generation over Slow Fading Channels," 2021 IEEE International Symposium on Information Theory (ISIT), Melbourne, Australia, 2021.

[3] R. Ezzine, M. Wiese, C. Deppe and H. Boche, "A General Formula for Uniform Common Randomness Capacity," 2022 IEEE Information Theory Workshop (ITW), Mumbai, India, 2022.

[4] R. Ezzine, M. Wiese, C. Deppe and H. Boche, "A Lower and Upper Bound on the Epsilon-Uniform Common Randomness Capacity," 2023 IEEE International Symposium on Information Theory (ISIT), Taipei, Taiwan, 2023.

CR Generation with One-Way Communication

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CR Generation with One-Way Communication

Source Type	Channel Type	Known Result
finite	Gaussian	Single-letter CR capacity Formula[1]
finite	Slow Fading with arbitrary state distribution	Single-letter outage CR capacity Formula[2]
finite	Arbitrary	General Formula/Bounds on the Uniform CR Capacity[3]/ϵ-Uniform CR Capacity[4].

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CR Generation from Compound Sources

- **Compound source**: the source derives from a defined uncertainty set, and the latter remains unchanged during the observation time scale
- The concept of generating shared randomness from compound sources was introduced solely in the context of secret key generation[1][2][3]
- No secrecy requirements are imposed in our work

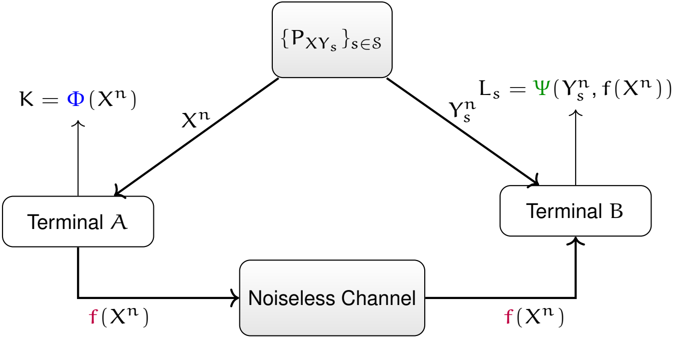
[1] H. Boche and R. F. Wyrembelski, "Secret key generation using compound sources - optimal key-rates and communication costs," in 9th International ITG Conference on Systems, Communication and Coding, pp. 1-6, 2013.

[2] N. Tavangaran, S. Baur, A. Grigorescu and H. Boche, "Compound Biometric Authentication Systems with Strong Secrecy," in 11th International ITG Conference on Systems, Communications and Coding, pp. 1-5, 2017.

[3] N. Tavangaran, H. Boche and R. F. Schaefer, Secret-Key Generation Using Compound Sources and One-Way Public Communication," *IEEE Transactions on Information Forensics and Security*, vol. 12, no. 1, pp. 227-241, 2017.

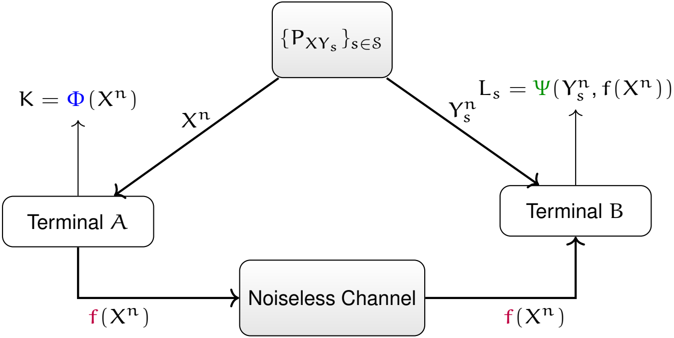
Outline

System Model



- $\{P_{XY_s}\}_{s\in\mathcal{S}}$ is a compound discrete memoryless multiple source (CDMMS)
- $\mathcal{S}$: Finite set of source states
- both terminals know the set of source states as well as their statistics with probability distributions $\{P_{XY_s}\}_{s\in\mathcal{S}}$ but they don't know the actual state $s \in \mathcal{S}$

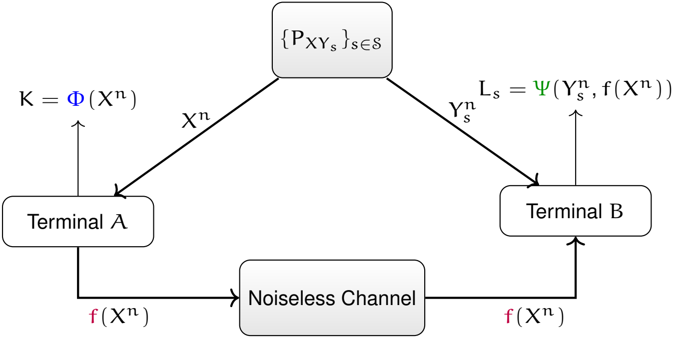
System Model



- The CDMMS emits i.i.d. samples of (X, Y_s)
- Communication over a noiseless channel with capacity $R > 0$

No other resources are available!

System Model



- (K, L_s) is called permissible pair
- **Question**: How much CR can we generate in one-way communication such that $\mathbb{P}[K \neq L_s]$ is small for every $s \in \mathcal{S}$?

Achievable Compound CR Rate

A number H is called an achievable compound CR rate if for every $\alpha, \delta > 0$ and for sufficiently large n there exists a permissible pair of random variables (K, L_s) for every $s \in \mathcal{S}$ such that

$$\forall s \in \mathcal{S} : \mathbb{P}[K \neq L_s] \leqslant \alpha$$

and

$$\frac{1}{n} H(K) > \mathrm{H} - \delta$$

Compound CR Capacity

The compound CR capacity $C_{CCR}(R)$ is the maximum achievable compound CR rate

Outline

Lower Bound on the Compound CR Capacity

Theorem

The compound CR capacity $C_{CCR}(R)$ *satisfies*

$$C_{CCR}(R) \geqslant \max_{\substack{U \\ \forall s \in \mathcal{S}: U \multimap X \multimap Y_s \\ I(U;X) - \min_{s \in \mathcal{S}} I(U;Y_s) \leqslant R}} I(U;X)$$

- *The alphabet* $\mathcal{U}$ *of* U *is subject to the cardinality bound*

$$|\mathcal{U}| \leqslant |\mathcal{X}| + |\mathcal{S}| - 1$$

Upper Bound on the Compound CR Capacity

Theorem

The compound CR capacity $C_{CCR}(R)$ *satisfies*

$$C_{CCR}(R) \leqslant \min_{s \in \mathcal{S}} \max_{\substack{U_s \\ U_s \multimap X \multimap Y_s \\ I(U_s;X) - I(U_s;Y_s) \leqslant R}} I(U_s;X)$$

- *For every* $s \in \mathcal{S}$, *the alphabet* $\mathcal{U}_s$ *of* U_s *is subject to the cardinality bound*

$$|\mathcal{U}_s| \leqslant |\mathcal{X}|$$

Facts

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- If there exists $s' \in \mathcal{S}$ such that for every $s \in \mathcal{S}$, $X \multimap Y_s \multimap Y_{s'}$ forms a Markov chain, then, the bounds are equal
- For $R < \max_{s \in \mathcal{S}} H(X|Y_s)$,

$$\max_{\substack{U \\ \forall s \in \mathcal{S}: U \multimap X \multimap Y_s \\ I(U;X) - \min_{s \in \mathcal{S}} I(U;Y_s) \leqslant R}} I(U;X) \geqslant R$$

The lower-bound is not tight. **Example:** Consider $\mathcal{S} = \{s_0\}$ and $X = Y_{s_0}$ then for all $R > 0$

$$\max_{\substack{U \\ \forall s \in \mathcal{S}: U \multimap X \multimap Y_s \\ I(U;X) - \min_{s \in \mathcal{S}} I(U;Y_s) \leqslant R}} I(U;X) = H(X)$$

Outline

Proof Sketch of the Lower Bound

Extend the scheme provided by Ahslwede and Csiszár [1] to compound sources:

- Generation of nearly uniform random variables K and L_s for every $s \in \mathcal{S}$, s.t for sufficiently large n such that for every $\alpha > 0$

$$\forall s \in \mathcal{S} : \mathbb{P}\left[K \neq L_s\right] \leqslant \alpha \checkmark$$

$$\frac{1}{n}H(K) > I(U;X) - \delta \checkmark$$

for any U satisfying $\forall s \in \mathcal{S} : U \multimap X \multimap Y_s$ and $I(U;X) - \min_{s \in \mathcal{S}} I(U;Y_s) < R$

[1] R. Ahlswede and I. Csiszár, "Common randomness in information theory and cryptography. II. CR capacity," in IEEE Transactions on Information Theory, vol. 44, no. 1, pp. 225-240, Jan. 1998.

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2. Consider a set of sequences $\mathbf{U}_{i,j}, i = 1 \dots N_1, j = 1 \dots N_2$, uniformly distributed on $\mathcal{T}_\sigma^n(P_U)$ and let $\mathbf{u}_{i,j}$ some realization of $\mathbf{U}_{i,j}, i = 1 \dots N_1, j = 1 \dots N_2$,

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3. $x^n \xrightarrow{\Phi} \mathbf{u}_{i,j} : (\mathbf{u}_{i,j}, x^n) \in \mathcal{T}_\sigma^n(P_{UX})$

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4. $(y_s^n, i) \xrightarrow{\Psi} \mathbf{u}_{i,\tilde{j}}$:
   - The index i is sent over the noiseless channel
   - $y_s^n \xrightarrow{E_i} \tilde{j}$ if there exists $s^\star \in \mathcal{S}$ such that $(\mathbf{u}_{i,\tilde{j}}, y_s^n) \in \mathcal{T}_\sigma^n(P_{UY_{s^\star}})$
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     - $\mathcal{T}_\sigma^n(P_{UY_{s^\star}})$: Set of σ-strongly typical sequences with respect to $P_{UY_{s^\star}}(\cdot)$

No state estimation is required at Terminal B!

Proof Sketch of the Lower Bound

- Proof of the cardinality bound $|\mathcal{U}| \leqslant |\mathcal{X}| + |\mathcal{S}| - 1$ based on a direct application of the support lemma

Support Lemma [1]

Let $\mathcal{P}(X)$ be the family of all probability distributions on the set X, and let f_j, $j = 1, \ldots, k$ be real-valued continuous functions on $\mathcal{P}(X)$. Then to any probability measure μ on the Borel σ-algebra of $\mathcal{P}(X)$ there exist k elements P_i of $\mathcal{P}(X)$ and non-negative numbers $\alpha_1, \ldots, \alpha_k$ with $\sum_{i=1}^{k} \alpha_i = 1$ such that for every $j = 1, \ldots, k$:

$$\int_{\mathcal{P}(X)} f_j(P)\, \mu(dP) = \sum_{i=1}^{k} \alpha_i f_j(P_i)$$

[1] I. Csiszár and J. Körner, *Information Theory: Coding Theorems for Discrete Memoryless Systems*," 2nd ed. Cambridge University Press, 2011.

Proof Sketch of the Upper Bound

We prove that for every $s \in \mathcal{S}: \frac{H(K)}{n} \leqslant \max_{\substack{U_s \\ U_s \circ\!\!-\, X \circ\!\!-\, Y_s \\ I(U_s;X) - I(U_s;Y_s) \leqslant R + \zeta(n)}} I(U_s; X)$, for

$\zeta(n) > 0$ where $\lim_{n \to \infty} \zeta(n)$ can be made arbitrarily small ✓

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   - $U_s \multimap X_J \multimap Y_{s,J}$
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3. Show that $\frac{H(K|Y_s^n)}{n} = I(U_s;X_J) - I(U_s;Y_{s,J})$

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- Proof of the cardinality bound $|\mathcal{U}_s| \leqslant |\mathcal{X}|$ is based on a direct application of the support lemma.

Outline

Conclusions

- ☺ We established a single-letter lower and upper bound on the compound CR capacity for our proposed model
- ☺ We considered two special scenarios where the established bounds coincide
- ☺ **Open Problem**: Derivation of a single-letter formula of the compound CR capacity for our system model

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