Characterization of the Complexity of Computing the Capacity of Colored Noise Gaussian Channels

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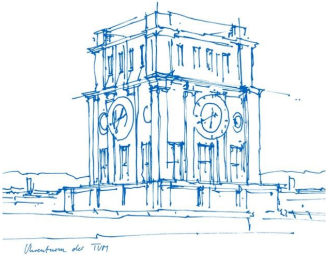

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Outline

Motivation

- 6G expand Tactile Internet and IoT for consumers
- 6G central requirement: **trustworthines**[1],[2]
- Find good codes operating at high rates
- How efficient are codes?
- Is it possible to improve efficiency?

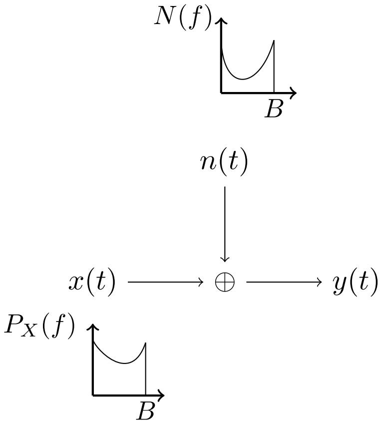
⇒ **Quantify capacity using current technology**

[1] G. P. Fettweis and H. Boche, "On 6G and trustworthiness," *Commun. ACM*, vol. 65, no. 4, pp. 48–49, Apr. 2022.

[2] G. P. Fettweis and H. Boche, "6G: The personal Tactile Internet—and open questions for information theory," *IEEE BITS Inf. Theory Mag.*, vol. 1, no. 1, pp. 71–82, Sep. 2021.

Band-limited ACGN Channel

- Band $B > 0$
- $x(t)$ Band-limited input signal with p.s.d. $P_X(f)$
- $y(t)$ Band-limited output signal
- $n(t)$ Band-limited Gaussian noise with noise p.s.d. $N(f)$



Capacity of ACGN Channels[3]

Theorem 1

The capacity of the band-limited ACGN channel with bandwidth B, and continuous noise spectrum N on the interval $[0, B]$ subject to a power constraint $P > 0$ is given by

$$C(N, P) = \int_0^B \ln\left(1 + \frac{P_x^*(f)}{N(f)}\right) df.$$

The capacity-achieving power spectral density is given by

$$P_x^*(f) = \begin{cases} \left[\nu - N(f)\right]_+ & \text{for } f \in [0, B] \\ 0 & f \notin [0, B], \end{cases}$$

where ν is chosen such that $\int_0^B P_x^*(f)\, df = P$ is satisfied.

ACGN capacity formula looks simple, however...

[3] C. E. Shannon, "Communication in the presence of noise," *Proc. IRE*, vol. 37, no. 1, pp. 10–21, 1949.

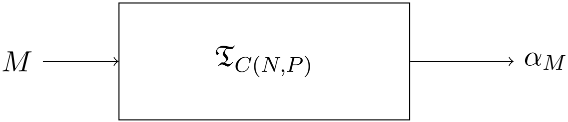
Computability ACGN Channel Capacity

There are computable noise spectrum, for which the capacity yields a **non-computable number**[4]
$\Rightarrow$ Shannon's coding approach is not effective, i.e., cannot be solved algorithmically

Is it possible to restrict the set of band-limited ACGN channels such that the capacity becomes computable for this set of channels?

[4] H. Boche, A. Grigorescu, R. F. Schaefer, and H. V. Poor, "Algorithmic computability of the capacity of Gaussian channels with colored noise," in *Proc. IEEE Global Telecommun. Conf.*, Kuala Lumpur, Malaysia, Dec. 2023, pp. 4375–4380.

Computability ACGN Capacity



How can the set of noise power spectra be **restricted** *such that for each N in this subset and a fixed P, it is* **possible to find an algorithm** *that takes a precision M as input and computes the number $\alpha(N, M)$ with*

$$|C(N,P) - \alpha(N,M)| < \frac{1}{2^M}?$$

Computability of ACGN Channel Capacity

Theorem 2

If N is a strictly positive and computable continuous noise p.s.d. and $P \in \mathbb{Q}$ with $P > \hat{P}$ and $\hat{P} = f(B, \bar{N}, \max N(f))$, then the capacity $C(P, N) \in \mathbb{R}_c$.[5]

Question 1

What is the computational complexity of computing the capacity $C(P, N)$?

[5] H. Boche, A. Grigorescu, R. F. Schaefer, and H. V. Poor, "Characterization of the complexity of computing the capacity of colored noise Gaussian channels," *IEEE Trans. Commun.*, 2024, Early Access.

Computational Complexity

Definition 1 (Class FP)

A function $f\colon \{0,1\}^* \to \mathbb{N}$ is in FP if it can be computed by a deterministic TM in polynomial time

Definition 2 (Class $\#\mathrm{P}$)

A function $f\colon \{0,1\}^* \to \mathbb{N}$ is in $\#\mathrm{P}$ if there exists a polynomial $p\colon \mathbb{N} \to \mathbb{N}$ and a polynomial time TM M, such that for every string $x \in \{0,1\}^*$,

$$f(x) = |\{y \in \{0,1\}^{p(|x|)} \colon M(x,y) = 1\}|$$

Comparision Complexity Classes

Decision problem

- **P**: decision problems solvable in polynomial time
- **NP**: decision problems verifiable in polynomial time

Counting Problem

- **FP**: counting functions computable in polynomial time
- **#P**: functions that count verifiable solutions in polynomial time

Computational Complexity of ACGN Channel Capacity

Theorem 3

Let B be a polynomial time computable number, and N be a strictly positive and polynomial time computable noise power specturm. Then the computation of the capacity $C(P, N)$ for $P \in \mathbb{Q}$, $P > \bar{N}B$ is in $\#\mathrm{P}_1$. Furthermore, there is an infinitely differentiable and strictly positive N_* and a $P_* > \bar{N}_* B$ where $\bar{N}_*$ is the average noise p.s.d., such that the computation of $C(P_*, N_*)$ **cannot be polynomial time computable** *if $\mathrm{FP}_1 \neq \#\mathrm{P}_1$.*[6]

Complexity Blowup!

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[6] H. Boche, A. Grigorescu, R. F. Schaefer, and H. V. Poor, "Characterization of the complexity of computing the capacity of colored noise Gaussian channels," *IEEE Trans. Commun.*, 2024, Early Access.

Implications of Complexity Blowup

Finite blocklength regime

- Let $\{R_n(\epsilon)\}_{n\in\mathbb{N}}$ blocklength-dependent sequence of achievable rates when allowing error $\epsilon \in \mathbb{Q}$
- $\{R_n(\epsilon)\}_{n\in\mathbb{N}}$ converges to the capacity
  - Either the sequence of achievable rates is not a polynomial-time computable sequence
  - Or the minimum blocklength n_M corresponding to the capacity approximation

$$R_{n_M}(\epsilon) > C(P, N_*) - \frac{1}{2^M}$$

grows faster than any polynomial[7]

[7] H. Boche, A. Grigorescu, R. F. Schaefer, and H. V. Poor, "Finite blocklength performance of capacity-achieving codes in the light of complexity theory," in *IEEE Int. Symp. Inf. Theory*, Recent Result Sess., IEEE, 2024.

Computational Complexity

Shannon's capacity formula and the power allocation formula are usually considered "closed formulas", but is this really correct?

If this naive approach were correct, then the answer to the following question would be very easy to find!

Question 2

What is the computational complexity of computing the values of the capacity-achieving input p.s.d. $P_x^*(P, \cdot)$ as a function of the frequency?

Complexity of Capacity Achieving p.s.d.

Theorem 4

Let $B \in \mathbb{R}_c$ be a polynomial time computable number, $N : [0, B] \to \mathbb{R}$ be a polynomial time computable continuous function and $P \in \mathbb{Q}$ with $P > \bar{N}B$ be arbitrary. Then $P_x^*(P, 0)$ is in $\#\mathrm{P}_1$. Furthermore, there exists a strictly positive computable noise power spectrum N_* that is infinitely differentiable on $[0, B]$, such that for all $P > \bar{N}_* B$, where $\bar{N}_*$ is the average noise power and $P \in \mathbb{Q}$, the function $\boldsymbol{P_x^*(P, 0)}$ **is complete in** $\#\mathbf{P_1}$.[8]

[8] H. Boche, A. Grigorescu, R. F. Schaefer, and H. V. Poor, "Characterization of the complexity of computing the capacity of colored noise Gaussian channels," *IEEE Trans. Commun.*, 2024, Early Access.

Conclusions

- If the continuous noise power spectrum is **strictly positive and computable**, then the capacity will always be a **computable number**.
- If $\mathrm{FP}_1 \neq \#\mathrm{P}_1$ holds true, then there exists a **polynomial time computable** N_*, whose **capacity** exhibits a **complexity blowup**.
- If $\mathrm{FP}_1 \neq \#\mathrm{P}_1$ holds true, then there exists a **polynomial time computable** N_*, whose **capacity-achieving p.s.d.** exhibits a **complexity blowup**.

Thank you for your attention!

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Literature

- G. P. Fettweis and H. Boche, “On 6G and trustworthiness,” *Commun. ACM*, vol. 65, no. 4, pp. 48–49, Apr. 2022
- G. P. Fettweis and H. Boche, “6G: The personal Tactile Internet—and open questions for information theory,” *IEEE BITS Inf. Theory Mag.*, vol. 1, no. 1, pp. 71–82, Sep. 2021
- C. E. Shannon, “Communication in the presence of noise,” *Proc. IRE*, vol. 37, no. 1, pp. 10–21, 1949
- H. Boche, A. Grigorescu, R. F. Schaefer, and H. V. Poor, “Algorithmic computability of the capacity of Gaussian channels with colored noise,” in *Proc. IEEE Global Telecommun. Conf.*, Kuala Lumpur, Malaysia, Dec. 2023, pp. 4375–4380
- H. Boche, A. Grigorescu, R. F. Schaefer, and H. V. Poor, “Characterization of the complexity of computing the capacity of colored noise Gaussian channels,” *IEEE Trans. Commun.*, 2024, Early Access
- H. Boche, A. Grigorescu, R. F. Schaefer, and H. V. Poor, “Finite blocklength performance of capacity-achieving codes in the light of complexity theory,” in *IEEE Int. Symp. Inf. Theory*, Recent Result Sess., IEEE, 2024