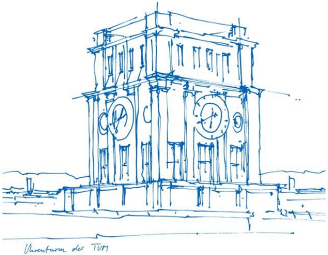
On the Solvability of Resource Allocation Problems for Wireless Systems on Digital Computers

Holger Boche[†], Andrea Grigorescu[†], Rafael F. Schaefer[*] and H. Vincent Poor[‡]

[†]Chair of Theoretical Information Technology, Technical University of Munich
[*] Chair of Information Theory and Machine Learning, Technische Universität Dresden
[‡] Department of Electrical and Computer Engineering, Princeton University



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Outline

Motivation

- 6G expand Tactile Internet and IoT for consumers
- 6G central requirement: **trustworthines**[1,2]
- Base-band signal processing implemented on digital processors

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[1] G. P. Fettweis and H. Boche, "On 6G and trustworthiness," *Commun. ACM*, vol. 65, no. 4, pp. 48–49, Apr. 2022.

[2] G. P. Fettweis and H. Boche, "6G: The personal Tactile Internet—and open questions for information theory," *IEEE BITS Inf. Theory Mag.*, vol. 1, no. 1, pp. 71–82, Sep. 2021.

Motivation

- Wireless communication relies on efficient resource allocation
- Network-centric and user-centric objectives
- Power constraints
- Interdependencies generated by interference $\Rightarrow$ jointly optimization of communication links

Given a communication network, find an algorithm that optimizes resource allocation problems subject to power constraint

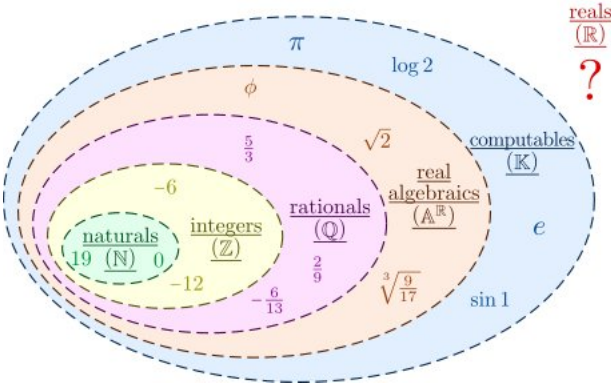
How can we formalize this in a precise and rigorous way?

Computability Framework[3]

- Alan M. Turing was the first to study this kind of problems systematically
- He developed a computing model
  - $\Rightarrow$ **Turing machines**
- Object of interest: real numbers

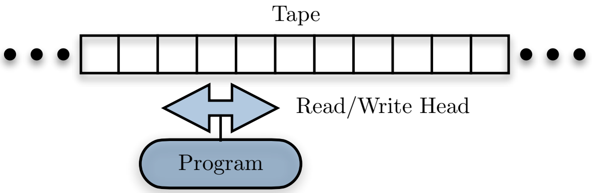
[3] A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proc. London Math. Soc.*, vol. 2, no. 42, pp. 230–265, Nov. 1936.

Computable Numbers



https://mathvoices.ams.org/featurecolumn/2021/12/01/alan-turing-computable-numbers/

Turing Machine[4]



- Mathematical model of an abstract machine that manipulates symbols on a strip of tape according to a table of rules
- Error free execution
- No limitations on computational complexity
- No limitation on computing capacity or storage

[4] A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proc. London Math. Soc.*, vol. 2, no. 42, pp. 230–265, Nov. 1936.

Computable Numbers

Definition 1

A sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ is called a *computable sequence* if there exist recursive functions $a, b, s : \mathbb{N} \to \mathbb{N}$ with $b(n) \neq 0$ for all $n \in \mathbb{N}$ and

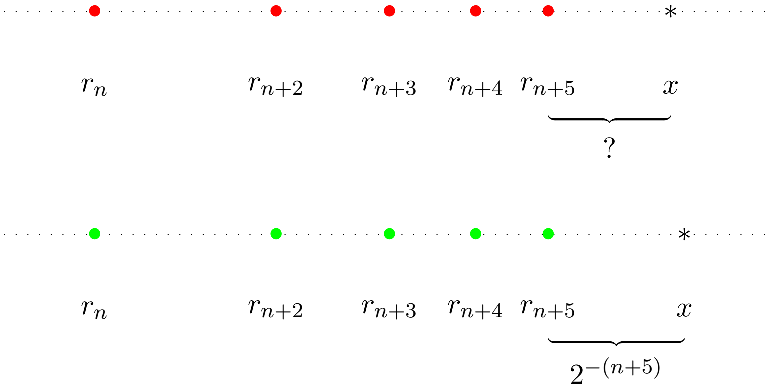
$$r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}, \qquad n \in \mathbb{N}$$

Definition 2

A real number x is said to be computable if there exists a computable sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$, such that

$$|x - r_n| < 2^{-n} \qquad n \in \mathbb{N}$$

Computable Numbers



Wireless Setup

- Wireless communication system with K parties
- Link between parties $V \in \mathbb{R}^{K \times K}$
- Link from user l to user k: $V_{k,l} \geq 0$ with $1 \leq k \leq K$ and $1 \leq l \leq K$
- Power allocation $P \in \mathbb{R}^{K}$
- SINR k-th receiver:

$$\gamma_k(P) = SINR_k(P) = \frac{\alpha_k P_k}{\sum_{\substack{l=1 \\ l \neq k}} V_{k,l} P_l + \sigma_k^2}$$

P_k power level k-th signal, $\sum_{l=1}^{K} V_{k,l} P_l$ accumulated interference power, $\sigma_k^2 > 0$ noise power, and $\alpha_k > 0$ matched filter effect on the receiver[5][6]

[5] H. Boche and M. Schubert, "Resource allocation in multiantenna systems-achieving max-min fairness by optimizing a sum of inverse sir," *IEEE Trans. Signal Process.*, vol. 54, no. 6, pp. 1990–1997, Jun. 2006.

[6] M. Schubert and H. Boche, *Interference calculus: A general framework for interference management and network utility optimization*. Springer Science & Business Media, 2011, vol. 7.

Power Constraint

- Individual power constraint: $\mathcal{M}_{\text{ind}}(\Psi, \lambda) = \{P \in \mathbb{R}^K \colon P \geq 0, \Psi(P_k) \leq \lambda\}$.
- Total power constraint: $\mathcal{M}_{\text{sum}}(\Psi, \lambda) = \{P \in \mathbb{R}^K \colon P \geq 0, \sum_k^K \Psi(P_k) \leq \lambda\}$

Standard formulation of **convex** optimization problems[7]

[7] S. P. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

Max-min Fairness[8],[9]

- The capacity with individual power constraint:

$$C^{\text{ind}}_{\text{maxmin}}(\Psi) = \max_{P \in \mathcal{M}_{\text{ind}}(\Psi)} \min_{1 \leq k \leq K} \gamma_k(P)$$

- The capacity with total power constraint:

$$C^{\text{sum}}_{\text{maxmin}}(\Psi) = \max_{P \in \mathcal{M}_{\text{sum}}(\Psi)} \min_{1 \leq k \leq K} \gamma_k(P)$$

Several algorithms were proposed to solve interference balancing. However, there is no general stopping criterion until now

Is it possible to find a general algorithmical stopping criterion?

[8] H. Boche and M. Schubert, "Concave and convex interference functions—general characterizations and applications," *IEEE Trans. Signal Process.*, vol. 56, no. 10, pp. 4951–4965, Oct. 2008.

[9] H. Boche and M. Schubert, "The structure of general interference functions and applications," *IEEE Trans. Inf. Theory*, vol. 54, no. 11, pp. 4980–4990, 2008.

Core Question

Question 1:

For a fixed individual power constraint function Ψ, is it possible to find an algorithm that takes the parameters α_k, V, and σ^2 as input and computes the max-min fairness $C^{\mathit{ind}}_{\mathit{maxmin}}(\alpha, V, \sigma^2)$? For the same parameters, is it possible to find an algorithm that computes the max-min power allocation?

Question 2:

For a fixed total power constraint function Ψ, is it possible to find an algorithm that takes the parameters α_k, V, and σ^2 as input and computes the max-min fairness $C^{\mathit{sum}}_{\mathit{maxmin}}(\alpha, V, \sigma^2)$? For the same parameters, is it possible to find an algorithm that computes the max-min power allocation?

Computability Max-min Fairness

Theorem 1

There is a computable convex function Ψ, such that for every computable σ_k^2 and computable V with we have that

1. *For every optimal power allocation $\hat{P}$ of the max-min fairness problem under the sum constraint, the following holds: If $V_{k,k} = 0$, for $1 \leq k \leq K$, then $\hat{P}$ contains* **non-computable elements** *and $C^{\mathit{sum}}_{\mathit{maxmin}}(\Psi)$ is* **not a computable number**
2. *For every optimal vector $\tilde{P}$ of the max-min fairness problem under individual power constraints it holds that the vector $\tilde{P}$* **non-computable elements**. *It also holds that $C^{\mathit{ind}}_{\mathit{maxmin}}(\Psi)$ is* **not a computable number**[10]

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[10] H. Boche, A. Grigorescu, R. F. Schaefer, and H. V. Poor, "On the solvability of resource allocation problems for wireless systems on digital computers," in *Proc. IEEE Int. Conf. Commun.*, to appear, 2024.

Computability Max-min Fairness

- There is a convex and computable power constraint function (sum & ind), such that there is no stopping criterion for either approximating the max-min fairness or approximating the max-min power allocation at any desired precision
- There is **no algorithm** for either the max-min fairness or the max-min fair allocation that can take any precision and stops at the desired precision
- **There is no general stopping criterion for SNIR Balancing!**

Utility Maximization[11],[12]

- Gain function: $\varphi\colon [0,\infty) \to \mathbb{R}$
- Total gain: $\sum_{k=1}^{K} \beta_k \varphi(\gamma_k(P))$
- Weight importance k-th party: $\beta_k \geq 0$, $1 \leq k \leq K$

$$C^{\text{ind}}_{\text{utility}}(\Psi) = \max_{P \in \mathcal{M}_{\text{ind}}(\Psi)} \sum_{k=1}^{K} \beta_k \varphi(\gamma_k(P))$$

$$C^{\text{sum}}_{\text{utility}}(\Psi) = \max_{P \in \mathcal{M}_{\text{sum}}(\Psi)} \sum_{k=1}^{K} \beta_k \varphi(\gamma_k(P)).$$

[11] H. Boche, S. Naik, and T. Alpcan, "Characterization of convex and concave resource allocation problems in interference coupled wireless systems," *IEEE Trans. Signal Process.*, vol. 59, no. 5, pp. 2382–2394, May 2011.

[12] T. Alpcan, H. Boche, M. L. Honig, and H. V. Poor, *Mechanisms and games for dynamic spectrum allocation*. Cambridge University Press, 2013.

Core Question

Question 3:

For a fixed individual power constraint function Ψ, is it possible to find an algorithm that takes the parameters α_k, V, and σ^2 as input and computes the optimal power allocation vector $\hat{P}(\alpha, V, \sigma^2)$ that maximizes the utility function $C_{\textit{utility}}^{\textit{ind}}(\Psi)$?

Question 4:

For a fixed total power constraint function Ψ, is it possible to find an algorithm that takes the parameters α_k, V, and σ^2 as input and computes the optimal power allocation vector $\tilde{P}(\alpha, V, \sigma^2)$ that maximizes the utility function $C_{\textit{utility}}^{\textit{sum}}(\Psi)$?

Computability Power Allocation Utility

Theorem 2

There is a computable convex function Ψ, such that for every computable σ_k^2 and computable V such that

1. *Every optimal power allocation $\hat{P} \in \mathcal{M}_{sum}$ such that*

$$C_{utility}^{sum}(\Psi) = \sum_{k=1}^{K} \beta_k \phi(\gamma_k(\hat{P}))$$

   contains elements that are not in $\mathbb{R}_c$

2. *Every optimal power allocation $\tilde{P} \in \mathcal{M}_{ind}$ such that*

$$C_{utility}^{ind}(\Psi) = \sum_{k=1}^{K} \beta_k \phi(\gamma_k(\tilde{P}))$$

   the vector $\tilde{P}$ contains elements that are not $\mathbb{R}_c$[a]

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[a] H. Boche, A. Grigorescu, R. F. Schaefer, and H. V. Poor, "On the solvability of resource allocation problems for wireless systems on digital computers," in *Proc. IEEE Int. Conf. Commun.*, to appear, 2024.

Computability Power Allocation Utility

- There is **no algorithm** that, when given a precision of $\frac{1}{2^M}$, stops when the computed power allocation is within a margin of error of $\frac{1}{2^M}$ from the optimal power allocation

Conclusions

- There exists a computable convex constraint function, both for individual and total power, such that for every computable noise, computable communication link matrix, and computable gain functions, the optimal power allocation vector for utility maximization proves to contain **non-computable elements.**
- Max-min fairness scenario under the same power constraint, both max-min fairness and the corresponding power allocation vector contain **non-computable elements**.

Thank you for your attention!

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