Time-Varying Systems and Computations

Unit 9.2

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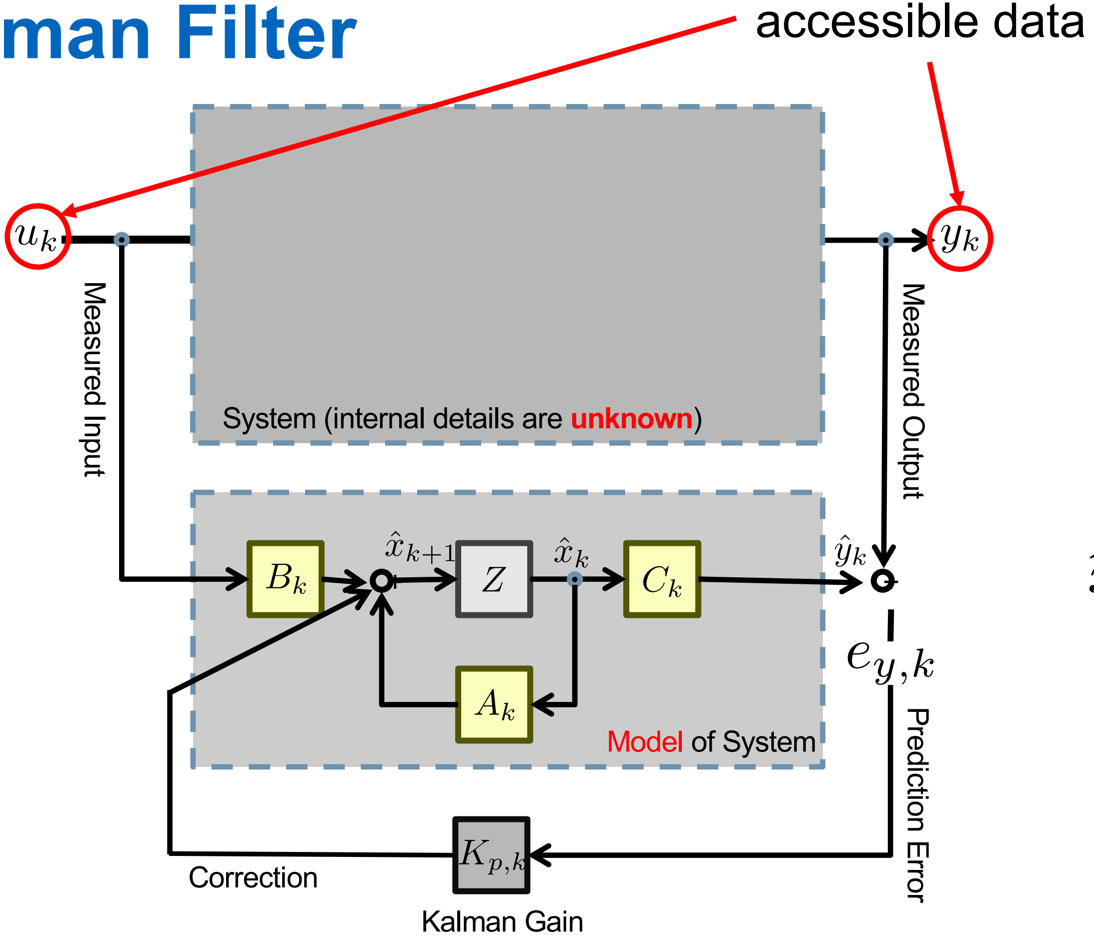
Kalman Filter



$$\widehat{y}_k = C_k \widehat{x}_k$$

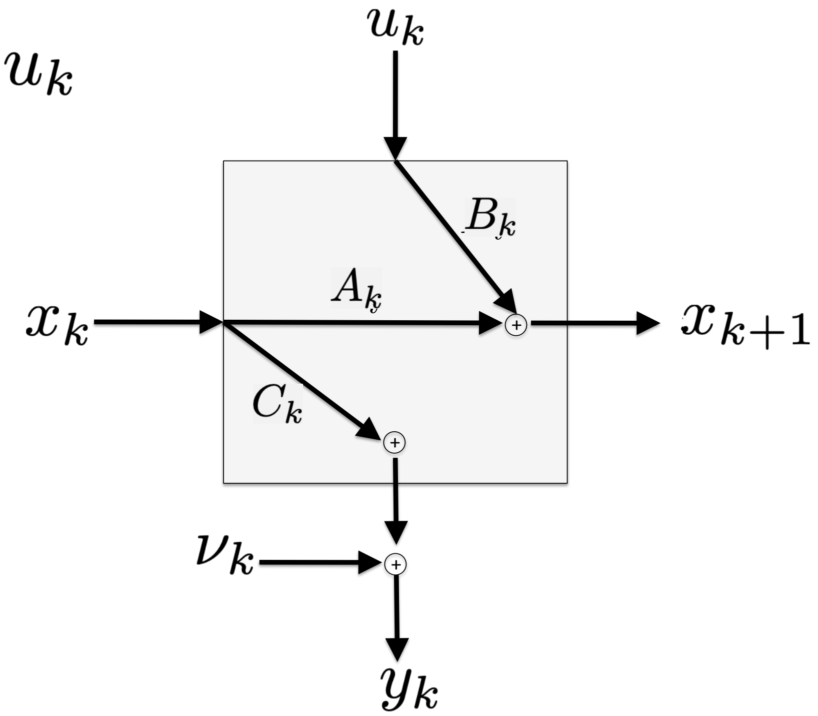
$$e_{y,k} = y_k - \widehat{y}_k$$
$$= y_k - C_k \widehat{x}_k$$

$$\widehat{x}_{k+1} = A_k \widehat{x}_k + B_k u_k$$
$$+ K_{p,k}(y_k - C\widehat{x}_k)$$

Kalman Filter revisited

- State Equations

$$\begin{cases} x_{k+1} & = & A_k x_k + B_k u_k \\ y_k & = & C_k x_k + \nu_k \end{cases}$$



- Estimate

state vectors	$\widehat{x}_{k+1}$
based on outputs	y_k
to minimize error	$e_{y,k}$

Covariance Matrix

- Stochastic variables

$$\mathbf{E}\{uy^T\} = \left\{ \begin{bmatrix} - & u_1 & - \\ \vdots & \vdots & \vdots \\ - & u_m & - \end{bmatrix} \begin{bmatrix} | & \cdots & | \\ y_1 & \cdots & y_n \\ | & \cdots & | \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \mathbf{E}\{u_1 y_1^T\} & \mathbf{E}\{u_1 y_2^T\} & \cdots & \mathbf{E}\{u_1 y_n^T\} \\ \mathbf{E}\{u_2 y_1^T\} & \mathbf{E}\{u_2 y_2^T\} & \cdots & \mathbf{E}\{u_2 y_n^T\} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{E}\{u_m y_1^T\} & \mathbf{E}\{u_m y_2^T\} & \cdots & \mathbf{E}\{u_m y_n^T\} \end{bmatrix}$$

Covariances

Covariances of input, state and noise signals (known à priori)

$$\mathbf{E}\{x_k x_k^T\} = \Pi_k \qquad \mathbf{E}\{x_0 x_0^T\} = \Pi_0$$

$$\mathbf{E}\{u_k u_k^T\} = Q_k \qquad \mathbf{E}\{\nu_k \nu_k^T\} = R_k$$

Error signals – prediction errors for output and state vector

$$e_{y,k} = y_k - \widehat{y}_k \qquad e_{x,k} = x_k - \widehat{x}_k$$

Covariance of prediction errors

$$R_{e_{y,k}} = \mathbf{E}\{e_{y,k} e_{y,k}^T\} \quad R_{e_{x,k}} = \mathbf{E}\{e_{x,k} e_{x,k}^T\}$$

Kalman Updates

Prediction   Correction

State prediction

$$\widehat{x}_{k+1} = A_k \widehat{x}_k + K_{p,k} \underbrace{(y_k - C_k \widehat{x}_k)}_{e_{y,k}}$$

Measurement Prediction Error

$$K_{p,k} = K_k R_{e,k}^{-1}$$ Kalman Gain

$$R_{e,k} = E\{e_{y,k} e_{y,k}^T\}$$

$$K_k = A_k P_k C_k^T \quad R_{e,k} = R_k + C_k P_k C_k^T$$

$$P_k = E\{e_{x,k} e_{x,k}^T\}$$

$$K_{p,k} = A_k P_k C_k^T (R_k + C_k P_k C_k^T)^{-1}$$

Update for covariance

$$P_{k+1} = A_k P_k A_k^T + B_k Q_k B_k^T - K_{p,k} R_{e,k} K_{p,k}^T$$

Root of all evil

Initial State $\hat{x}_0 = 0, \ P_0 = \Pi_0$

Square Root Kalman (LQ-type)

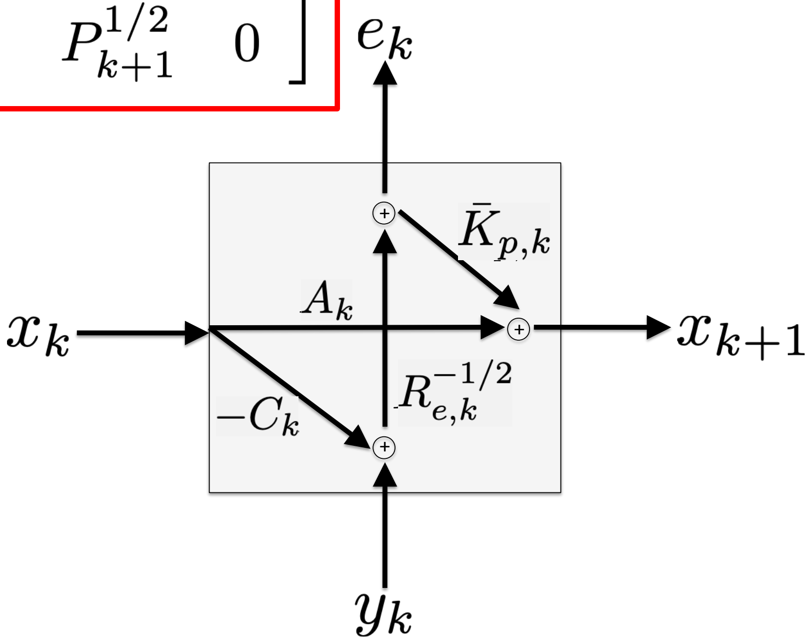
$$\begin{bmatrix} C_k P_k^{1/2} & R_k^{1/2} & 0 \\ A_k P_k^{1/2} & 0 & B_k Q_k^{1/2} \end{bmatrix} U_k = \begin{bmatrix} R_{e,k}^{1/2} & 0 & 0 \\ \bar{K}_{p,k} & P_{k+1}^{1/2} & 0 \end{bmatrix}$$

$$P_k = P_k^{1/2} P_k^{T/2}$$

Outer filter → causally invertible

$$\begin{bmatrix} A_k & \bar{K}_{p,k} \\ C_k & R_{e,k}^{1/2} \end{bmatrix}$$

$$\bar{K}_{p,k} = K_{p,k} R_{e,k}^{1/2} = K_k R_{e,k}^{-T/2}$$



Squaring the Left Side

Orthogonal matrix → U_k

$$\begin{bmatrix} C_k P_k^{1/2} & R_k^{1/2} & 0 \\ A_k P_k^{1/2} & 0 & B_k Q_k^{1/2} \end{bmatrix} U_k U_k^T \begin{bmatrix} P_k^{T/2} C_k^T & P_k^{T/2} A_k^T \\ R_k^{T/2} & 0 \\ 0 & Q_k^{T/2} B_k^T \end{bmatrix}$$

$$= \begin{bmatrix} C_k P_k C_k^T + R_k & C_k P_k A_k^T \\ A_k P_k C_k^T & A_k P_k A_k^T + B_k Q_k B_k^T \end{bmatrix}$$

Squaring the Right Side

$$\begin{bmatrix} R_{e,k}^{1/2} & 0 & 0 \\ \bar{K}_{p,k} & P_{k+1}^{1/2} & 0 \end{bmatrix} \begin{bmatrix} R_{e,k}^{T/2} & \bar{K}_{p,k}^{T} \\ 0 & P_{k+1}^{T/2} \\ 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} R_{e,k} & R_{e,k}\bar{K}_{p,k}^{T} \\ \bar{K}_{p,k}R_{e,k}^{T/2} & \bar{K}_{p,k}\bar{K}_{p,k}^{T} + P_{k+1} \end{bmatrix}$$

Equating Squares

$$\begin{bmatrix} C_k P_k C_k^T + R_k & C_k P_k A_k^T \\ A_k P_k C_k^T & A_k P_k A_k^T + B_k Q_k B_k^T \end{bmatrix}$$

left side

$$= \begin{bmatrix} R_{e,k} & K_k \\ K_k & K_{p,k} R_{e,k} K_{p.k}^T + P_{k+1} \end{bmatrix}$$

right side

Kalman Filter Recursion

$$P_{k+1} = A_k P_k A_k^T + B_k Q_k B_k^T - K_{p,k} R_{e,k} K_{p,k}^T$$

$$K_k = A_k P_k C_k^T \qquad R_{e,k} = R_k + C_k P_k C_k^T$$

$$K_{p,k} = K_{p,k} R_{e,k}^{1/2} = K_k R_{e,k}^{-T/2}$$

Numerically stable computation

Kalman@LDV