Time-Varying Systems and Computations

Unit 8.1

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Task Description

Transfer Function for causal system

$$T = D + C(I - ZA)^{-1}ZB$$

Compute Factorization

$$T = T_0 V$$

T_0 Causal, causally (left) invertible → <u>Outer Factor</u>

V Causal, unitary → <u>Inner Factor</u>

→ Inner-Outer Factorization

Inner-Outer Factorization

- Computational Task

  Recursive Computation of Factorization

  QR Factorization computed directly in State-Space

- Inner Factor

  $V'V = VV' = 1$ Unitary, causal

- Outer Factor

  T_o Causal (lower triangular), invertible

Bases for Range of Transfer Matrix

- Compute Basis for Range and Kernel

$$T = \begin{bmatrix} T_o & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$col(T_o)$ Basis for Range of T

$row(V_1)$ Orthogonal basis for co-range of T

$row(V_2)$ Orthogonal basis for kernel of T

Prototype 4x4-Example

Matrix given in terms of State-Space Realization

$$T = \begin{bmatrix} \boxed{D_0} & & & \\ C_1 B_0 & D_1 & & \\ C_2 A_1 B_0 & C_2 B_1 & D_2 & \\ C_3 A_2 A_1 B_0 & C_3 A_2 B_1 & C_3 B_2 & D_3 \end{bmatrix}$$

Compute Factorization in State-Space

$$T = \begin{bmatrix} T_o & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = T_0 V_1$$

Prototype 4x4-Example

- Compute RQ Factorization of Part of

$$T = \begin{bmatrix} \boxed{D_0} & & & \\ C_1 B_0 & D_1 & & \\ C_2 A_1 B_0 & C_2 B_1 & D_2 & \\ C_3 A_2 A_1 B_0 & C_3 A_2 B_1 & C_3 B_2 & D_3 \end{bmatrix}$$

- Compute RQ Factorization of Part of T

$$\begin{bmatrix} B_0 \\ D_0 \end{bmatrix} := \begin{bmatrix} 0 & Y_1 & B_{o0} \\ 0 & 0 & D_{o0} \end{bmatrix} [Q_1]$$

Prototype 4x4-Example

$$\begin{bmatrix} B_0 \\ D_0 \end{bmatrix} := \begin{bmatrix} 0 & Y_1 & B_{o0} \\ 0 & 0 & D_{o0} \end{bmatrix} [Q_1]$$

- Insert Factorization Result

$$\left[\begin{array}{c|ccc} \begin{bmatrix} 0 & 0 & D_{o0} \end{bmatrix} & & & \\ C_1 \begin{bmatrix} 0 & Y_1 & B_{o0} \end{bmatrix} & D_1 & & \\ C_2 A_1 \begin{bmatrix} 0 & Y_1 & B_{o0} \end{bmatrix} & C_2 B_1 & D_2 & \\ C_3 A_2 A_1 \begin{bmatrix} 0 & Y_1 & B_{o0} \end{bmatrix} & C_3 A_2 B_1 & C_3 B_2 & D_3 \end{array}\right]$$

Prototype 4x4-Example

$$\left[\begin{array}{c|ccc} \begin{bmatrix} 0 & 0 & D_{o0} \end{bmatrix} & & & \\ C_1 \begin{bmatrix} 0 & Y_1 & B_{o0} \end{bmatrix} & D_1 & & \\ C_2 A_1 \begin{bmatrix} 0 & Y_1 & B_{o0} \end{bmatrix} & C_2 B_1 & D_2 & \\ C_3 A_2 A_1 \begin{bmatrix} 0 & Y_1 & B_{o0} \end{bmatrix} & C_3 A_2 B_1 & C_3 B_2 & D_3 \end{array}\right]$$

- Expand Submatrices

$$\left[\begin{array}{ccc|ccc} 0 & 0 & D_{o0} & & & \\ 0 & C_1 Y_1 & C_1 B_{o0} & D_1 & & \\ 0 & C_2 A_1 Y_1 & C_2 A_1 B_{o0} & C_2 B_1 & D_2 & \\ 0 & C_3 A_2 A_1 Y_1 & C_3 A_2 A_1 B_{o0} & C_3 A_2 B_1 & C_3 B_2 & D_3 \end{array}\right]$$

Prototype 4x4-Example

$$\left[\begin{array}{cccc|ccc} 0 & 0 & D_{o0} & & & \\ 0 & C_1Y_1 & C_1B_{o0} & D_1 & & \\ 0 & C_2A_1Y_1 & C_2A_1B_{o0} & C_2B_1 & D_2 & \\ 0 & C_3A_2A_1Y_1 & C_3A_2A_1B_{o0} & C_3A_2B_1 & C_3B_2 & D_3 \end{array}\right]$$

- Sort Columns

$$\left[\begin{array}{cccc|ccc} 0 & D_{o0} & 0 & & & \\ 0 & C_1B_{o0} & C_1Y_1 & D_1 & & \\ 0 & C_2A_1B_{o0} & C_2A_1Y_1 & C_2B_1 & D_2 & \\ 0 & C_3A_2A_1B_{o0} & C_3A_2A_1Y_1 & C_3A_2B_1 & C_3B_2 & D_3 \end{array}\right]$$

Prototype 4x4-Example

$$\left[\begin{array}{cccc|ccc} 0 & D_{o0} & 0 & & & \\ 0 & C_1 B_{o0} & C_1 Y_1 & D_1 & & \\ 0 & C_2 A_1 B_{o0} & C_2 A_1 Y_1 & C_2 B_1 & D_2 & \\ 0 & C_3 A_2 A_1 B_{o0} & C_3 A_2 A_1 Y_1 & C_3 A_2 B_1 & C_3 B_2 & D_3 \end{array}\right]$$

- Extract Submatrix for Factorization

$$\left[\begin{array}{cc|cc} C_1 Y_1 & D_1 & 0 & 0 \\ \hline C_2 A_1 Y_1 & C_2 B_1 & D_2 & \\ C_3 A_2 A_1 Y_1 & C_3 A_2 B_1 & C_3 B_2 & D_3 \end{array}\right]$$

Prototype 4x4-Example

$$\left[\begin{array}{cc|cc} C_1Y_1 & D_1 & 0 & 0 \\ \hline C_2A_1Y_1 & C_2B_1 & D_2 & \\ C_3A_2A_1Y_1 & C_3A_2B_1 & C_3B_2 & D_3 \end{array}\right]$$

- Compute RQ Factorization of Submatrix

$$\begin{bmatrix} A_1Y_1 & B_1 \\ C_1Y_1 & D_1 \end{bmatrix} = \begin{bmatrix} 0 & Y_2 & B_{o1} \\ 0 & 0 & D_{o1} \end{bmatrix} Q_2$$

- Insert Factorization Result

$$\begin{bmatrix} 0 & 0 & D_{o1} \\ 0 & C_2Y_2 & C_2B_{o1} \\ 0 & C_3A_2Y_2 & C_3A_2B_{o1} \end{bmatrix}$$

Prototype 4x4-Example

- Sort Columns

$$\left[\begin{array}{cc|cc|ccc} 0 & 0 & D_{o0} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_1 B_{o0} & D_{o1} & 0 & 0 & 0 \\ 0 & 0 & C_2 A_1 B_{o0} & C_2 B_{o1} & C_2 Y_2 & D_2 & 0 \\ 0 & 0 & C_3 A_2 A_1 B_{o0} & C_3 A_2 B_{o1} & C_3 A_2 Y_2 & C_3 B_2 & D_3 \end{array}\right]$$

- Compute RQ Factorization of Submatrix

$$\begin{bmatrix} A_2 Y_2 & B_2 \\ C_2 Y_2 & D_2 \end{bmatrix} = \begin{bmatrix} 0 & Y_3 & B_{o2} \\ 0 & 0 & D_{o2} \end{bmatrix} \cdot Q_3$$

Prototype 4x4-Example

- Insert Factorization into

$$\left[\begin{array}{cc|cc|ccc} 0 & 0 & D_{o0} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_1 B_{o0} & D_{o1} & 0 & 0 & 0 \\ 0 & 0 & C_2 A_1 B_{o0} & C_2 B_{o1} & C_2 Y_2 & D_2 & 0 \\ 0 & 0 & C_3 A_2 A_1 B_{o0} & C_3 A_2 B_{o1} & C_3 A_2 Y_2 & C_3 B_2 & D_3 \end{array}\right]$$

- … and after reshuffling results in

$$\left[\begin{array}{ccc|ccc|cc} 0 & 0 & 0 & D_{o0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_1 B_{o0} & D_{o1} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 A_1 B_{o0} & C_2 B_{o1} & D_{o2} & 0 & 0 \\ 0 & 0 & 0 & C_3 A_2 A_1 B_{o0} & C_3 A_2 B_{o1} & C_3 B_{o2} & C_3 Y_3 & D_3 \end{array}\right]$$

Prototype 4x4-Example

- Extract Submatrix

$$\left[\begin{array}{ccc|ccc|cc} 0 & 0 & 0 & D_{o0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_1 B_{o0} & D_{o1} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 A_1 B_{o0} & C_2 B_{o1} & D_{o2} & 0 & 0 \\ 0 & 0 & 0 & C_3 A_2 A_1 B_{o0} & C_3 A_2 B_{o1} & C_3 B_{o2} & C_3 Y_3 & D_3 \end{array}\right]$$

- Compute Factorization of Submatrix

$$\left[\begin{array}{cc} C_3 Y_3 & D_3 \end{array}\right] = \left[\begin{array}{cc} 0 & D_{o3} \end{array}\right] \cdot Q_4$$

Prototype 4x4-Example

- Inserting factorization result into

$$\left[\begin{array}{ccc|cccc|cc} 0 & 0 & 0 & D_{o0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_1 B_{o0} & D_{o1} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 A_1 B_{o0} & C_2 B_{o1} & D_{o2} & 0 & 0 \\ 0 & 0 & 0 & C_3 A_2 A_1 B_{o0} & C_3 A_2 B_{o1} & C_3 B_{o2} & C_3 Y_3 & D_3 \end{array}\right]$$

- Produces the result (Outer Factor)

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & D_{o0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_1 B_{o0} & D_{o1} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_2 A_1 B_{o0} & C_2 B_{o1} & D_{o2} & 0 \\ 0 & 0 & 0 & 0 & C_3 A_2 A_1 B_{o0} & C_3 A_2 B_{o1} & C_3 B_{o2} & D_{o3} \end{array}\right]$$

Finish of Prototype 4x4-Example

- Outer Factor

$$T_o = D_o + C(I - ZA)^{-1} ZB_o \qquad \Sigma_o = \left[\begin{array}{c|c} A & B_o \\ \hline C & D_o \end{array}\right]$$

- Pseudo-Inverse of Outer Factor

$$T_o^+ = D_o^+ - CD_o^+(I - Z\Delta)^{-1} ZD_o^+ B_o \qquad \Delta = A - B_o D_o^+ C$$

- Pseudo-Inverse of Matrix

$$T^+ = V' \cdot T_o^+$$

- Inner Factor (product of orthogonal matrices)

$$V = \hat{Q}_1 \cdot \hat{Q}_2 \cdot \hat{Q}_3 \cdot \hat{Q}_4$$