Time-Varying Systems and Computations

Unit 7.3

Klaus Diepold

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Orthogonal Realization Matrix

- Orthogonal realization matrix (block diagonal) $\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with

$$\Sigma^T \Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^T \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} 1 & \\ \hline & 1 \end{array}\right]$$

- Properties of the realization matrix entries

$$B^T B = 1 - D^T D \qquad C^T C = 1 - A^T A \qquad B^T A = -D^T C$$

- → Transfer operator Q is orthogonal (inner)

$$Q = D - C(1 - ZA)^{-1} ZB \qquad \Rightarrow Q^T Q = 1$$

Orthogonal Realization

- Orthogonal realization Σ → orthogonal (inner) transfer operator Q

- Show that $1 - Q^T Q = 0$ for $\Sigma^T \Sigma = 1$
- Check

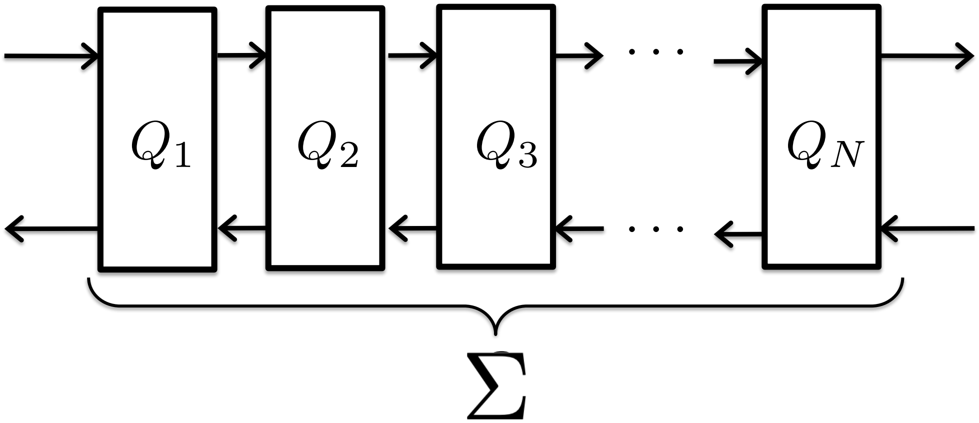
$$1 - \left(D - C(1 - ZA)^{-1} ZB\right)^T \left(D - C(1 - ZA)^{-1} ZB\right) =$$

[… after some lengthy but straight forward algebraic manipulations …]

$$= 0$$

Orthogonal Realization

- An orthogonal matrix – product of simple orthogonal transformations
- Reflections and rotations are simple orthogonal transformations



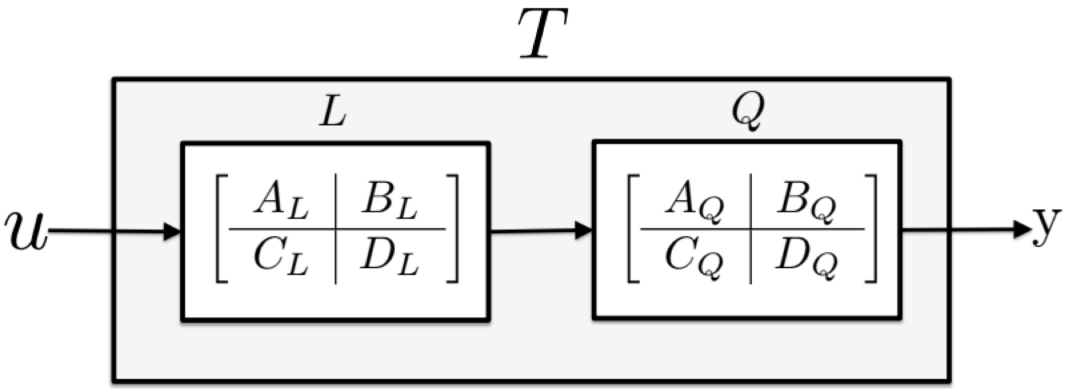
- Realization for an orthogonal transfer operator can be composed of rotations and reflections

What is it good for?

- QR-type factorization of matrix T

$$T = QL$$

- Perform factorization directly in state-space representation
- → Inner-Outer factorization



Detailed Calculation

$$
\begin{aligned}
0 &= 1 - \left[D + C(1_n - ZA)^{-1}ZB\right]^T \cdot \left[D + C(1_n - ZA)^{-1}ZB\right] \\
&= 1 - \left[D^T + B^T Z^T (1_n - A^T Z^T)^{-1} C^T\right] \cdot \left[D + C(1_n - ZA)^{-1}ZB\right] \\
&= 1 - D^T D - D^T C(1_n - ZA)^{-1}ZB - B^T Z^T (1_n - A^T Z^T)^{-1} C^T D \ldots \\
&\quad -B^T Z^T (1_n - A^T Z^T)^{-1} C^T C (1_n - ZA)^{-1} ZB \\
&= B^T B + B^T A (1_n - ZA)^{-1} ZB + B^T Z^T (1_n - A^T Z^T)^{-1} A^T B \ldots \\
&\quad -B^T Z^T (1_n - A^T Z^T)^{-1} (1 - A^T A)(1_n - ZA)^{-1} ZB \\
&= B^T Z^T (1 - A^T Z^T)^{-1} \left[(1 - A^T Z^T)(1 - ZA) + (1 - A^T Z^T) ZA \ldots \right. \\
&\quad \left. A^T Z^T (1 - ZA) - (1 - A^T A)\right] (1 - ZA)^{-1} ZB \\
&= B^T Z^T (1 - A^T Z^T)^{-1} \left[1 - A^T Z^T - ZA + A^T Z^T ZA + ZA - A^T Z^T ZA \ldots \right. \\
&\quad \left. + A^T Z^T - A^T Z^T ZA - 1 + A^T A\right] (1 - ZA)^{-1} ZB.
\end{aligned}
$$