Time-Varying Systems and Computations

Unit 6.3

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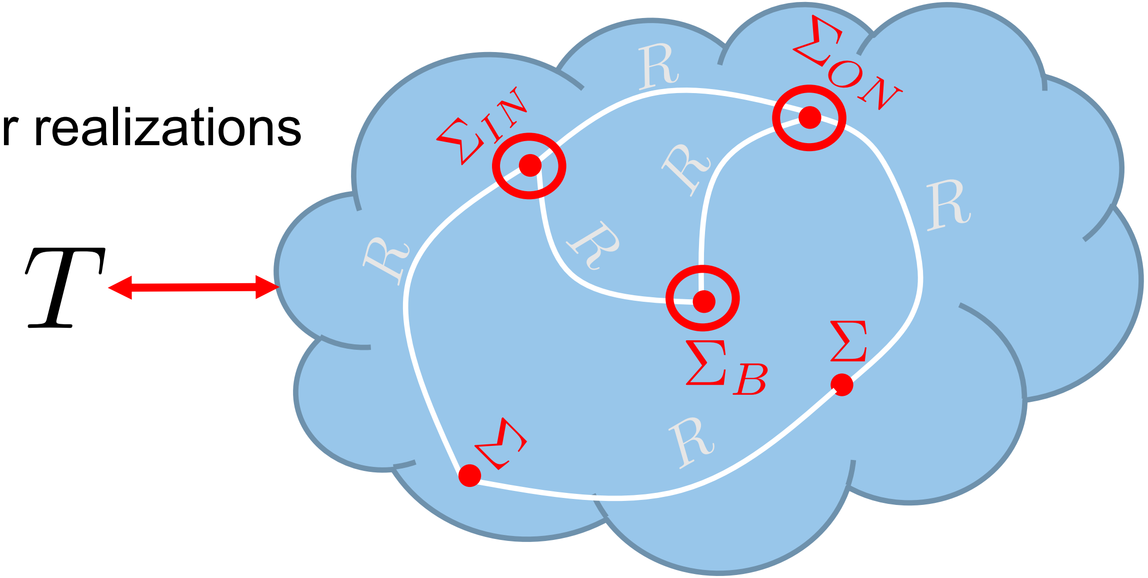
Particular state space realizations

- Infinite number of realizations $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ for one T
- All realizations are connected by state equivalence transformations
- Starting from one realization all others are parametrized by R

- Search for three particular realizations
  1. **Input Normal** Σ_{IN}
  2. **Output Normal** Σ_{ON}
  3. **Balanced** Σ_B



Input Normal Realization

- Orthogonal Reachability Gramian – orthogonal basis for input space

$$\mathcal{W}_k = \mathcal{R}_k \mathcal{R}_k^T = 1$$

- Determined by SVD-based factorization

$$\mathcal{H}_k = \mathcal{O}_k \mathcal{R}_k = (U_k \Sigma_k) V_k^T \quad \Rightarrow \quad \mathcal{O}_k = U_k \Sigma_k \quad \mathcal{R}_k = V_k^T$$

- Determined by QR-based factorization

$$\mathcal{H}_k = \mathcal{O}_k \mathcal{R}_k = L_k Q_k \quad \Rightarrow \quad \mathcal{O}_k = L_k \quad \mathcal{R}_k = Q_k$$

Input Normal Realization

- Determine Input Normal Realization starting from any realization
- Take Reachability Gramian for a given realization and compute Cholesky factorization

$$\mathcal{W}_k = \mathcal{R}_k \mathcal{R}_k^T = M_k M_k^T$$

- Apply state transformation using the Cholesky factor

$$\widehat{\mathcal{R}}_k = M_k^{-1} \mathcal{R}_k$$

- Resulting realization is input normal $\widehat{\mathcal{R}}_k \widehat{\mathcal{R}}_k^T = 1$

Output Normal Realization

- Orthogonal Reachability Gramian – orthogonal basis for output space

$$\mathcal{K}_k = \mathcal{O}_k^T \mathcal{O}_k = 1$$

- Determined by SVD-based factorization

$$\mathcal{H}_k = \mathcal{O}_k \mathcal{R}_k = U_k(\Sigma_k V_k^T) \quad \Rightarrow \quad \mathcal{O}_k = U_k \quad \mathcal{R}_k = \Sigma_k V_k^T$$

- Determined by QR-based factorization

$$\mathcal{H}_k = \mathcal{O}_k \mathcal{R}_k = Q_k R_k \quad \Rightarrow \quad \mathcal{O}_k = Q_k \quad \mathcal{R}_k = R_k$$

Output Normal Realization

- Determine Input Normal Realization starting with any realization
- Take Reachability Gramian for a given realization and compute Cholesky factorization

$$\mathcal{K}_k = \mathcal{O}_k^T \mathcal{O}_k = N_k^T N_k$$

- Apply state transformation using the Cholesky factor

$$\widehat{\mathcal{O}}_k = \mathcal{O}_k N_k^{-1}$$

- Resulting realization is output normal $\widehat{\mathcal{O}}_k^T \widehat{\mathcal{O}}_k = 1$

Balanced Realization

- Orthogonal Reachability Gramian

$$\mathcal{O}_k^T \mathcal{O}_k = \mathcal{R}_k \mathcal{R}_k^T = \Sigma_k$$

- Determined by SVD-based factorization

$$\mathcal{H}_k = \mathcal{O}_k \mathcal{R}_k = U_k \Sigma_k V_k^T \quad \Rightarrow \quad \mathcal{O}_k = U_k \sqrt{\Sigma_k} \quad \mathcal{R}_k = \sqrt{\Sigma_k} V_k^T$$

- QR-based factorization → doesn‘t work

$$\sqrt{\Sigma_k} = \begin{bmatrix} \sqrt{\sigma_1} & & & \\ & \sqrt{\sigma_2} & & \\ & & \ddots & \\ & & & \sqrt{\sigma_n} \end{bmatrix}$$

Balanced Realization

- Determine Balanced Realization starting with any realization

- Simultaneous Diagonalization to determine necessary state transformation

$$\mathcal{K}_k = \mathcal{O}_k^T \mathcal{O}_k \xrightarrow{M}$$

$$\mathcal{W}_k = \mathcal{R}_k \mathcal{R}_k^T \xrightarrow{N}$$

$$\mathcal{W}_k = \mathcal{K}_k = \Sigma_k$$

- → Advanced Topic … maybe next time

Determine Balancing Transformation

$$\mathcal{W}_k = \mathcal{R}_k \mathcal{R}_k^T = M_k M_k^T \qquad \mathcal{K}_k = \mathcal{O}_k^T \mathcal{O}_k = N_k^T N_k$$

$$N_k M_k = U_k \Sigma_k V_k^T$$

$$S_k = \Sigma_k^{-1/2} U_k^T N_k$$

$$\widehat{\mathcal{W}}_k = S_k \mathcal{W}_k S_k^{-1} = \Sigma_k^{-1/2} U_k^T N_k \mathcal{W}_k N_k^{-1} U_k \Sigma_k^{1/2}$$