Time-Varying Systems and Computations

Unit 6.2

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The story so far …

- Reachability Matrix

$$\mathcal{R}_k = \begin{bmatrix} B_{k-1} & A_{k-1}B_{k-2} & A_{k-1}A_{k-2}B_{k-3} & \dots \end{bmatrix}$$

- Observability Matrix

$$\mathcal{O}_k = \begin{bmatrix} C_k \\ C_{k+1}A_k \\ C_{k+2}A_{k+1}A_k \\ \vdots \end{bmatrix}$$

Gramian matrices

- Geometric properties for basis vectors collected as columns in X
  - Dimension of space
  - Angles between spanning vectors

- Gramian matrix $\mathcal{G}(X) = X^T X$

  - dimension of space spanned by columns of X $\quad \text{rank}\,(\mathcal{G}(X)) = n$
  - Euclidean length $\quad x_i^T x_i = \mathcal{G}_{i,i}$
  - Angles $\quad \cos\varphi(x_i, x_j) = \mathcal{G}_{i,j}$
  - Symmetric, positive semi-definite $\quad \mathcal{G}(X) = \mathcal{G}^T(X) \quad \mathcal{G}(X) \geq 0$

Observability Gramian

- For a given realization → Lyapunov-Stein equations

$$\mathcal{K}_k = \mathcal{O}_k^T \mathcal{O}_k$$

$$= \begin{bmatrix} C_k^T & A_k^T C_{k+1}^T & A_k^T A_{k+1}^T C_{k+2}^T & \cdots \end{bmatrix} \begin{bmatrix} C_k \\ C_{k+1} A_k \\ C_{k+2} A_{k+1} A_k \\ \vdots \end{bmatrix}$$

$$= C_k^T C_k + A_k^T C_{k+1}^T C_{k+1} A_k + A_k^T A_{k+1}^T C_{k+2}^T C_{k+2} A_{k+1} A_k + \cdots$$

$$= C_k^T C_k + A_k^T \underbrace{\left[C_{k+1}^T C_{k+1} + A_{k+1}^T C_{k+2}^T C_{k+2} A_{k+1} + \ldots \right]}_{\mathcal{K}_{k+1}} A_k + \cdots$$

$$\mathcal{K}_k = C_k^T C_k + A_k^T \mathcal{K}_{k+1} A_k$$

LTI-Version: $\mathcal{K} = C^T C + A^T \mathcal{K} A$

Reachability Gramian

- For a given realization → Lyapunov-Stein equations

$$\mathcal{W}_k = \mathcal{R}_k \mathcal{R}_k^T$$

$$= \begin{bmatrix} B_{k-1} & A_{k-1}B_{k-2} & A_{k-1}A_{k-2}B_{k-3} & \cdots \end{bmatrix} \begin{bmatrix} B_{k-1}^T \\ B_{k-2}^T A_{k-1}^T \\ B_{k-3}^T A_{k-2}^T A_{k-1}^T \\ \cdots \end{bmatrix}$$

$$= B_{k-1}B_{k-1}^T + A_{k-1}B_{k-2}B_{k-2}^T A_{k-1}^T + A_{k-1}A_{k-2}B_{k-3}B_{k-3}^T A_{k-2}^T A_{k-1}^T + \dots$$

$$= B_{k-1}B_{k-1}^T + A_{k-1}\underbrace{\left[B_{k-2}B_{k-2}^T + A_{k-2}B_{k-3}B_{k-3}^T A_{k-2}^T + \cdots\right]}_{\mathcal{W}_{k-1}} A_{k-1}^T$$

$$\mathcal{W}_k = B_{k-1}B_{k-1}^T + A_{k-1}\mathcal{W}_{k-1}A_{k-1}^T$$

LTI-Version: $\mathcal{W} = BB^T + A\mathcal{W}A^T$

… also … Equivalence Transformation

- Transformed state-space realization for time index k

$$\widehat{\Sigma}_k = \left[\begin{array}{c|c} \widehat{A}_k & \widehat{B}_k \\ \hline \widehat{C}_k & \widehat{D}_k \end{array}\right] = \left[\begin{array}{c|c} S_{k+1} A_k S_k^{-1} & S_{k+1} B_k \\ \hline C S_k^{-1} & D_k \end{array}\right]$$

$$\det S_k \neq 0, \quad \forall k$$

- What is the impact of state transformations on observability and reachability?

Observability under equivalence trafo

- Observability matrix of transformed system

$$\widehat{\mathcal{O}}_k = \begin{bmatrix} (C_k S_k^{-1}) \\ (C_{k+1} S_{k+1}^{-1})(S_{k+1} A_k S_k^{-1}) \\ (C_{k+2} S_{k+2}^{-1})(S_{k+2} A_{k+1} S_{k+1}^{-1})(S_{k+1} A_k S_k^{-1}) \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} C_k S_k^{-1} \\ C_{k+1} A_k S_k^{-1} \\ C_{k+2} A_{k+1} A_k S_k^{-1} \\ \vdots \end{bmatrix} = \mathcal{O}_k S_k^{-1}$$

- Observability Gramian – congruence transformation

$$\widehat{\mathcal{K}}_k = \widehat{\mathcal{O}}_k^T \widehat{\mathcal{O}}_k = S_k^{-T} (\mathcal{O}_k^T \mathcal{O}_k) S_k^{-1} = S_k^{-T} \mathcal{K}_k S_k^{-1}$$

Reachability under equivalence trafo

- Reachability matrix of transformed system

$$\widehat{\mathcal{R}}_k =$$

$$= \left[\begin{array}{cccc} S_k B_{k-1} & (S_k A_{k-1} S_{k-1}^{-1})(S_{k-1} B_{k-2}) & (S_k A_{k-1} S_{k-1}^{-1})(S_{k-1} A_{k-2} S_{k-2}^{-1})(S_{k-2} B_{k-2}) & \dots \end{array} \right.$$

$$= \left[\begin{array}{cccc} S_k B_{k-1} & S_k A_{k-1} B_{k-2} & S_k A_{k-1} A_{k-2} B_{k-2} & \dots \end{array} \right]$$

$$= S_k \mathcal{R}_k$$

- Reachability Gramian – congruence transformation

$$\widehat{\mathcal{W}}_k = \widehat{\mathcal{R}}_k \widehat{\mathcal{R}}_k^T = S_k (\mathcal{R}_k \mathcal{R}_k^T) S_k^{-1} = S_k \mathcal{W}_k S_k^{-1}$$

Gramian matrices

- Minimal realization
  - Observability/Reachability Gramians have full rank
  - Observability/Reachability are symm. pos. def.

- Reachability Gramian
  - captures geometric properties of input-space
  - provides for a forward recursive computational scheme

- Observability Gramian
  - captures geometric propertiies of output-space
  - provides for a backward recursive computational scheme