Time-Varying Systems and Computations

Unit 6.1

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State-Space Equivalence

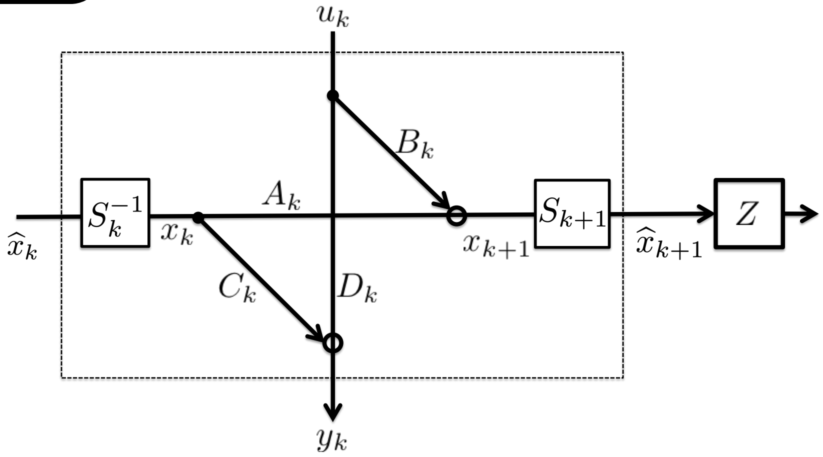
- Changing coordinates for state-space with non-singular matrix S_k

$$\widehat{x}_k := S_k x_k \qquad S_k^{-1}\widehat{x}_k = x_k$$

$$S_{k+1}^{-1}\widehat{x}_{k+1} = A_k S_k^{-1}\widehat{x}_k + B_k u_k$$

$$\widehat{x}_{k+1} = \underbrace{\left(S_{k+1} A_k S_k^{-1}\right)}_{\widehat{A}_k}\widehat{x}_k + \underbrace{\left(S_{k+1} B_k\right)}_{\widehat{B}_k} u_k$$

$$y_k = \underbrace{C_k S_k^{-1}}_{\widehat{C}_k}\widehat{x}_k + D_k u_k$$



Alternative Realization

- Transformed state-equations

$$\begin{bmatrix} \widehat{x}_{k+1} \\ y_k \end{bmatrix} = \left[\begin{array}{c|c} S_{k+1} & \\ \hline & 1 \end{array}\right] \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] \left[\begin{array}{c|c} S_k^{-1} & \\ \hline & 1 \end{array}\right] \begin{bmatrix} \widehat{x}_k \\ u_k \end{bmatrix}$$

- Transformed state-space realization for time index k

$$\widehat{\Sigma} = \left[\begin{array}{c|c} \widehat{A}_k & \widehat{B}_k \\ \hline \widehat{C}_k & \widehat{D}_k \end{array}\right] = \left[\begin{array}{c|c} S_{k+1} A_k S_k^{-1} & S_{k+1} B_k \\ \hline C_k S_k^{-1} & D_k \end{array}\right]$$

Alternative Realization

- Block diagonal form of time-varying state transformation matrix

$$S = \begin{bmatrix} \ddots & & \\ & \boxed{S_k} & \\ & & \ddots \end{bmatrix}$$

Diagonally shifted

$$S^{[-1]} = Z^{-1} S Z = \begin{bmatrix} \ddots & & \\ & \boxed{S_{k+1}} & \\ & & \ddots \end{bmatrix}$$

- State transformation - block diagonal style

$$\begin{bmatrix} Z^{-1}\widehat{x} \\ y \end{bmatrix} = \left[\begin{array}{c|c} S^{[-1]} & \\ \hline & 1 \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} S^{-1} & \\ \hline & 1 \end{array}\right] \begin{bmatrix} \widehat{x} \\ u \end{bmatrix}$$

Invariance of transfer operator

- Transfer operator after state transformation

$$\widehat{T} = \widehat{D} + \widehat{C}\left(1 - Z\widehat{A}\right)^{-1} Z\widehat{B}$$
$$= D + CS^{-1}\left(1 - ZS^{[-1]}AS^{-1}\right)^{-1} ZS^{[-1]}B$$
$$= T$$

- → Transfer operator / Toeplitz operator / impulse response is invariant under state transformation
- Every non-singular matrix S_k is permissible
  → infinite number of realizations for one given T (equivalence class)

Reaching all state space realizations

- Infinite number of realizations $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ for one T
- All equivalent realizations are connected through state transformations S
- Starting from one realization all others are parametrized by S
- Realizations exhibit different properties
- Search for particular realization
- → Optimization