Time-Varying Systems and Computations

Unit 5.3

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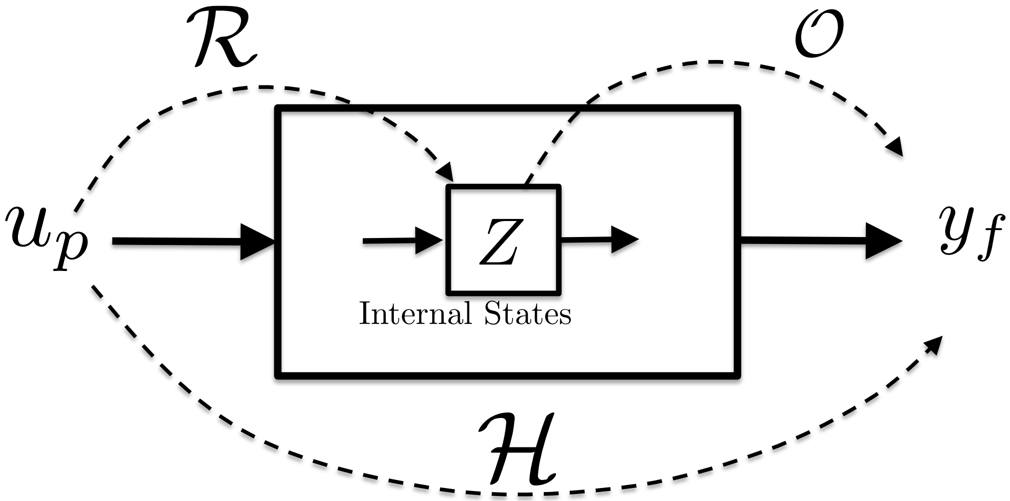
Factoring the Hankel Operator

- Factoring Hankel Operator

$$\mathcal{H}_k = \mathcal{O}_k \mathcal{R}_k$$

Observability

Reachability



Determine state space realization

- Observability/Reachability matrices contain the components of state space realization → directly extract C_k and B_{k-1} from $\mathcal{O}_k$ and $\mathcal{R}_k$

$$\mathcal{H}_k = \begin{bmatrix} C_k B_{k-1} & C_k A_{k-1} B_{k-2} & C_k A_{k-1} A_{k-2} B_{k-3} & \cdots \\ C_{k+1} A_k B_{k-1} & C_{k+1} A_k A_{k-1} B_{k-2} & C_{k+1} A_k A_{k-1} A_{k-2} B_{k-3} & \cdots \\ C_{k+2} A_{k+1} A_k B_{k-1} & C_{k+2} A_{k+1} A_k A_{k-1} B_{k-2} & C_{k+2} A_{k+1} A_k A_{k-1} A_{k-2} B_{k-3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} =$$

$$= \begin{bmatrix} C_k \\ C_{k+1} A_k \\ C_{k+2} A_{k+1} A_k \\ \vdots \end{bmatrix} \begin{bmatrix} B_{k-1} & A_{k-1} B_{k-2} & A_{k-1} A_{k-2} B_{k-3} & \cdots \end{bmatrix} = \mathcal{O}_k \mathcal{R}_k$$

- How can I access A_k?

Shift-invariance

- Invariance principle for **row space** of Hankel operator
  - Shifting the rows of $\mathcal{H}$ upwards – pre-multiplication with transposed shift operator

$$\mathrm{row}(Z^T\mathcal{H}) \subseteq \mathrm{row}(\mathcal{H})$$

- Invariance principle for **column space** of Hankel operator
  - Shifting the rows of $\mathcal{H}$ to the left – post-multiplication with shift operator

$$\mathrm{col}(\mathcal{H}Z) \subseteq \mathrm{col}(\mathcal{H})$$

Shift-invariance

Up-shifting and dropping first row → effect on observability matrix

$$\mathcal{H}\uparrow = \mathcal{P}_r Z' \mathcal{H}_k = \mathcal{O}_{k+1} A_k \mathcal{R}_k$$

$$\mathcal{H}_k\uparrow = \begin{bmatrix} C_{k+1}A_kB_{k-1} & C_{k+1}A_kA_{k-1}B_{k-2} & C_{k+1}A_kA_{k-1}A_{k-2}B_{k-3} & \cdots \\ C_{k+2}A_{k+1}A_kB_{k-1} & C_{k+2}A_{k+1}A_kA_{k-1}B_{k-2} & C_{k+2}A_{k+1}A_kA_{k-1}A_{k-2}B_{k-3} & \cdots \\ C_{k+3}A_{k+2}A_{k+1}A_kB_{k-1} & C_{k+3}A_{k+2}A_{k+1}A_kA_{k-1}B_{k-2} & C_{k+3}A_{k+2}A_{k+1}A_kA_{k-1}A_{k-2}B_{k-2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} C_{k+1} \\ C_{k+2}A_{k+1} \\ C_{k+3}A_{k+2}A_{k+1} \\ \vdots \end{bmatrix}}_{\mathcal{O}_{k+1}} \cdot A_k \cdot \underbrace{\begin{bmatrix} B_{k-1} & A_{k-1}B_{k-2} & A_{k-1}A_{k-2}B_{k-3} & \cdots \end{bmatrix}}_{\mathcal{R}_k}$$

Computing state-space realization

- Shifted Observability matrix $\mathcal{O}_k\!\uparrow = \mathcal{O}_{k+1} \cdot A_k$

- Compute matrix A_k from shifted Observability matrix

$$\Longrightarrow A_k = \mathcal{O}_{k+1}^{\dagger} \cdot \mathcal{O}_k\!\uparrow$$

$$B_k = \text{first column of } \mathcal{C}_{k+1}$$

$$C_k = \text{first row of } \mathcal{O}_k.$$

- … using Moore-Penrose Pseudo-Inverse

$$\mathcal{O}_{k+1}^{\dagger} = (\mathcal{O}_{k+1}^{T}\mathcal{O}_{k+1})^{-1}\mathcal{O}_{k+1}^{T}$$

- Similar steps based on left shifted Reachability matrix

Summary

- Factoring the Hankel Operator
- → Observability/Reachability Matrices

- Shifting Hankel Operator up or left leaves subspaces invariant

- Identification of state-space realization