Time-Varying Systems and Computations

Unit 5.2

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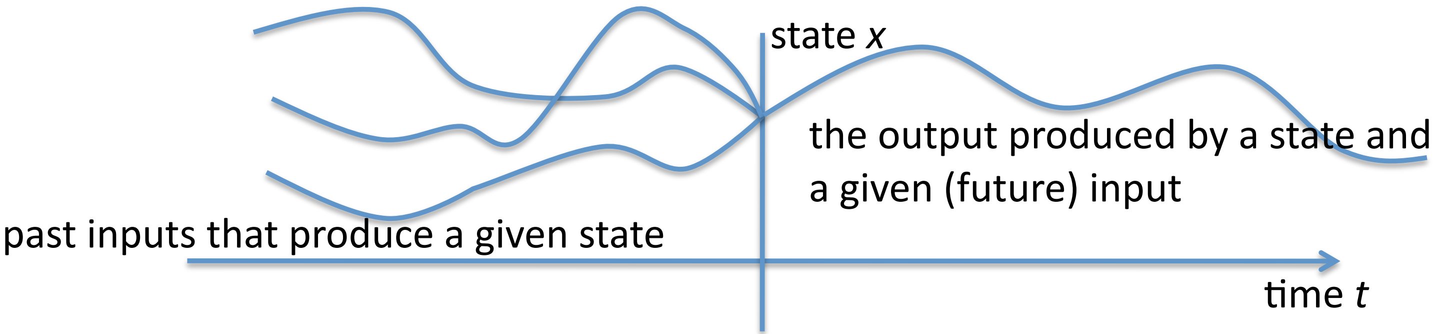
Hankel Operator

- Hankel Operator describes map from past input to future output
- We search for the state-space realization of T
- But … what is this Hankel Operator good for?

Nerode Equivalence



- Data in state vector x – stores relevant from the past

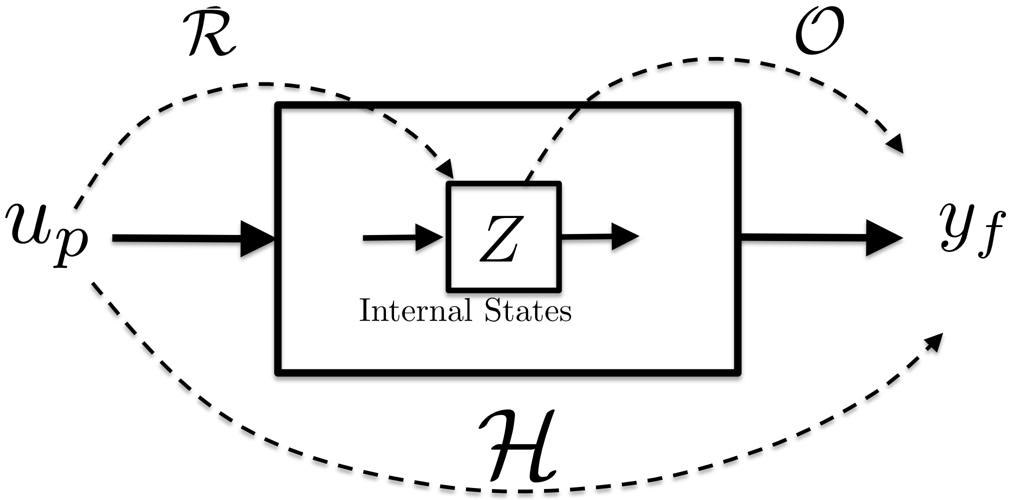
Factoring the Hankel Operator

- Splitting up the Hankel operator
  - $\mathcal{R}$ Input → Internal States
  - $\mathcal{O}$ Internal States → Output

- Factoring Hankel Operator

$$\mathcal{H}_k = \mathcal{O}_k \mathcal{R}_k$$

Observability Reachability



Reachability/Observability

- Reachability Matrix

$$\mathcal{R}_k = \begin{bmatrix} B_{k-1} & A_{k-1}B_{k-2} & A_{k-1}A_{k-2}B_{k-3} & \dots \end{bmatrix}$$

- Observability Matrix

$$\mathcal{O}_k = \begin{bmatrix} C_k \\ C_{k+1}A_k \\ C_{k+2}A_{k+1}A_k \\ \vdots \end{bmatrix}$$

Factoring the Hankel Operator

Hankel = Observability * Reachability =

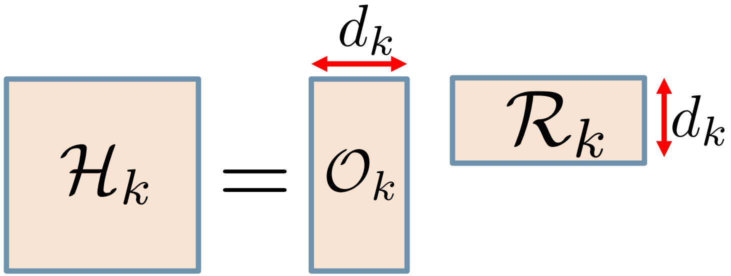
$$\begin{bmatrix} C_k B_{k-1} & C_k A_{k-1} B_{k-2} & C_k A_{k-1} A_{k-2} B_{k-3} & \cdots \\ C_{k+1} A_k B_{k-1} & C_{k+1} A_k A_{k-1} B_{k-2} & C_{k+1} A_k A_{k-1} A_{k-2} B_{k-3} & \cdots \\ C_{k+2} A_{k+1} A_k B_{k-1} & C_{k+2} A_{k+1} A_k A_{k-1} B_{k-2} & C_{k+2} A_{k+1} A_k A_{k-1} A_{k-2} B_{k-3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} =$$

$$= \begin{bmatrix} C_k \\ C_{k+1} A_k \\ C_{k+2} A_{k+1} A_k \\ \vdots \end{bmatrix} \begin{bmatrix} B_{k-1} & A_{k-1} B_{k-2} & A_{k-1} A_{k-2} B_{k-3} & \cdots \end{bmatrix}$$

Minimal Realizations

- Factorization is minimal

$$\mathcal{H}_k = \mathcal{O}_k \mathcal{R}_k$$



- Dimension of State Space – dynamic degree – number of latches

$$d_k = \text{rank}\ (\mathcal{H}_k)$$

- E.g. use the Singular Value Decomposition

$$\mathcal{H}_k = (U_k \Sigma_k) \cdot V_k^T$$

$$\Sigma_k = \left[\begin{array}{ccc|cc} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_{d_k} & & \\ \hline & & & 0 & \\ & & & & \ddots \end{array}\right]$$

Reachability/Observability

- Reachability Matrix with $\operatorname{rank}(\mathcal{R}_k) = d_k$ has d_k rows for all k
  → system is fully reachable

$$\mathcal{R}_k = \begin{bmatrix} B_{k-1} & A_{k-1}B_{k-2} & A_{k-1}A_{k-2}B_{k-3} & \dots \end{bmatrix}$$

- Observability Matrix with $\operatorname{rank}(\mathcal{O}_k) = d_k$ has d_k columns
  → system is fully observable

$$\mathcal{O}_k = \begin{bmatrix} C_k \\ C_{k+1}A_k \\ C_{k+2}A_{k+1}A_k \\ \vdots \end{bmatrix}$$

- Minimal realization is fully observable and fully reachable

Summary

- Partitioning the Hankel map → Factoring the Hankel Operator
- Observability/Reachability Matrices

- Rank of Hankel Operator determines dimension of state space („Kronecker Theorem“ – Leopold Kronecker)

- Minimal Realization is fully reachable and fully observable