Time-Varying Systems and Computations

Unit 5.1

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Realization Theory

- Collection of Time-Varying Impulse Responses → Convolution
- Given impulse response (Toeplitz)

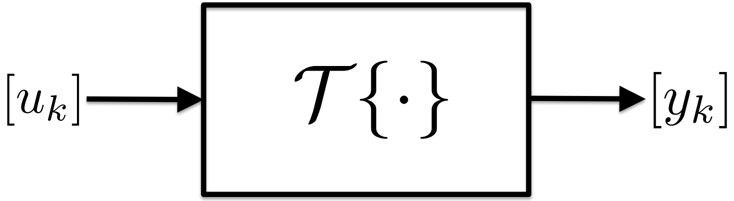
$$T = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & T_{-1,-1} & T_{-1,0} & T_{-1,1} & T_{-1,2} & \cdots \\ \cdots & T_{0,-1} & \boxed{T_{0,0}} & T_{0,1} & T_{0,2} & \cdots \\ \cdots & T_{1,-1} & T_{1,0} & T_{1,1} & T_{1,2} & \cdots \\ & \vdots & \vdots & \vdots & \cdots & \ddots \end{bmatrix}$$

→ how to determine a realization $\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

→ Transfer Function (Input-Output) $T = D + C(1 - ZA)^{-1}BZ$

Toeplitz Operator

- Time-Varying Input-Output Map - Convolution



$$T \cdot u = y$$

$$\left[\begin{array}{ccc|ccc} \ddots & \vdots & \vdots & \vdots & \vdots & \\ & T_{-2,-2} & T_{-2,-1} & T_{-2,0} & T_{-2,1} & \cdots \\ \cdots & T_{-1,-2} & T_{-1,-1} & T_{-1,0} & T_{-1,1} & \cdots \\ \hline \cdots & T_{0,-2} & T_{0,-1} & \boxed{T_{0,0}} & T_{0,1} & \cdots \\ \cdots & T_{1,-2} & T_{1,-1} & T_{1,0} & T_{1,1} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right] \left[\begin{array}{c} \vdots \\ u_{-2} \\ u_{-1} \\ \hline u_0 \\ u_1 \\ \vdots \end{array}\right] = \left[\begin{array}{c} \vdots \\ y_{-2} \\ y_{-1} \\ \hline y_0 \\ y_1 \\ \vdots \end{array}\right]$$

Restricting the Toeplitz Operator

- Past Input to Future Output

$$\begin{bmatrix} y_p \\ y_f \end{bmatrix} = \left[\begin{array}{c|c} T_{p,p} & T_{f,p} \\ \hline \underbrace{T_{p,f}}_{\mathcal{H}} & T_{f,f} \end{array}\right] \begin{bmatrix} u_p \\ u_f \end{bmatrix}$$

- Restrict Toeplitz Operator

$$\begin{bmatrix} \cdots & T_{0,-3} & T_{0,-2} & T_{0,-1} \\ \cdots & T_{1,-3} & T_{1,-2} & T_{1,-1} \\ \cdots & T_{2,-3} & T_{2,-2} & T_{2,-1} \\ & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ u_{-3} \\ u_{-2} \\ u_{-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix}$$

Restricting the Toeplitz Operator

- Restrict Toeplitz Operator

$$\begin{bmatrix} \cdots & T_{0,-3} & T_{0,-2} & T_{0,-1} \\ \cdots & T_{1,-3} & T_{1,-2} & T_{1,-1} \\ \cdots & T_{2,-3} & T_{2,-2} & T_{2,-1} \\ & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ u_{-3} \\ u_{-2} \\ u_{-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix}$$

- Flip the restricted Toeplitz Operator → Hankel Operator

$$\begin{bmatrix} T_{0,-1} & T_{0,-2} & T_{0,-3} & \cdots \\ T_{1,-1} & T_{1,-2} & T_{1,-3} & \cdots \\ T_{2,-1} & T_{2,-2} & T_{2,-3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} u_{-1} \\ u_{-2} \\ u_{-3} \\ \vdots \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix}$$

Restricting Toeplitz Operator

- Hankel Operator → Hankel Matrix in case of LTI systems

$$\begin{bmatrix} T_{0,-1} & T_{0,-2} & T_{0,-3} & \cdots \\ T_{1,-1} & T_{1,-2} & T_{1,-3} & \cdots \\ T_{2,-1} & T_{2,-2} & T_{2,-3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} u_{-1} \\ u_{-2} \\ u_{-3} \\ \vdots \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix}$$

LTI System

$$T_{i,j} \rightarrow T_{i-j}$$

Hankel Matrix

$$\begin{bmatrix} T_1 & T_2 & T_3 & \cdots \\ T_2 & T_3 & T_4 & \cdots \\ T_3 & T_4 & T_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} u_{-1} \\ u_{-2} \\ u_{-3} \\ \vdots \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix}$$

Hankel Operator

- Time-Varying Hankel Operator at time index $k = 0$

$$\mathcal{H}_0 = \begin{bmatrix} T_{0,-1} & T_{0,-2} & T_{0,-3} & \cdots \\ T_{1,-1} & T_{1,-2} & T_{1,-3} & \cdots \\ T_{2,-1} & T_{2,-2} & T_{2,-3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- Causality $\left[\begin{array}{c|c} T_{pp,k} & 0 \\ \hline \mathcal{H}_k & T_{ff,k} \end{array}\right] \begin{bmatrix} u_{p,k} \\ u_{f,k} \end{bmatrix} = \begin{bmatrix} y_{p,k} \\ y_{f,k} \end{bmatrix}$

- Construction of Time-Varying Hankel Operators $\mathcal{H}_k = ?$

Hankel Operator

- Time-Varying Hankel Operators (causal)

$$
\begin{array}{c} \\ \\ \\ \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \vdots \\ \\ \\ \\ \end{array}
\begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \\
\ddots & 0 & \vdots & \vdots & \\
\ddots & D_0 & 0 & \vdots & \\
\ddots & C_1 B_0 & D_1 & 0 & \\
\ddots & C_2 A_1 B_0 & C_2 B_1 & D_2 & \ddots \\
\ddots & C_3 A_2 A_1 B_0 & C_3 A_2 B_1 & C_3 B_2 & \ddots \\
 & C_4 A_3 A_2 A_1 B_0 & C_4 A_3 A_2 B_1 & C_4 A_3 B_2 & \ddots \\
 & \vdots & C_5 A_4 A_3 A_2 B_1 & C_5 A_4 A_3 B_2 & \ddots \\
 & \vdots & \vdots & C_6 A_5 A_4 A_3 B_2 & \ddots \\
 & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
$$

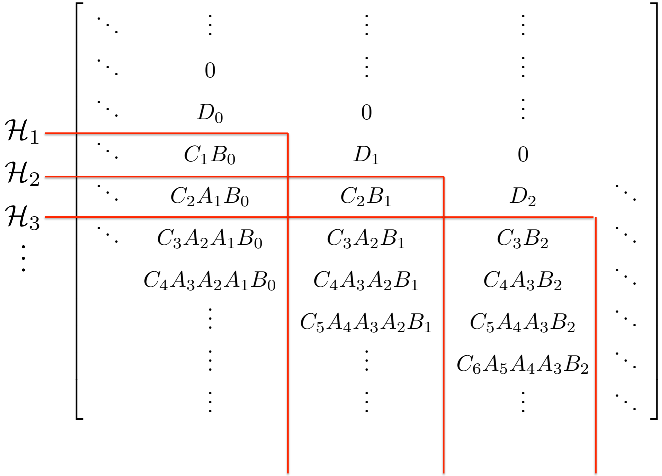
Sequence of Hankel Operators

- Read off the Time-Varying Hankel Operators

$$\mathcal{H}_1 = \begin{bmatrix} C_1B_0 & C_1A_0B_{-1} & C_1A_0A_{-1}B_{-2} & \cdots \\ C_2A_1B_0 & C_2A_1A_0B_{-1} & C_2A_1A_0A_{-1}B_{-2} & \cdots \\ C_3A_2A_1B_0 & C_3A_2A_1A_0B_{-1} & C_3A_2A_1A_0A_{-1}B_{-2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\mathcal{H}_2 = \begin{bmatrix} C_2B_1 & C_2A_1B_0 & C_2A_1A_0B_{-1} & \cdots \\ C_3A_2B_1 & C_3A_2A_1B_0 & C_3A_2A_1A_0B_{-1} & \cdots \\ C_4A_3A_2B_1 & C_4A_3A_2A_1B_0 & C_4A_3A_2A_1A_0B_{-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\mathcal{H}_3 = \begin{bmatrix} C_3B_2 & C_3A_2B_1 & C_3A_2A_1B_0 & \cdots \\ C_4A_3B_2 & C_4A_3A_2B_1 & C_4A_3A_2A_1B_0 & \cdots \\ C_5A_4A_3B_2 & C_5A_4A_3A_2B_1 & C_5A_4A_3A_2A_1B_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$



Hankel Operators

- Cut out – flip left right – general form of Hankel Operator

$$\mathcal{H}_k = \begin{bmatrix} T_{k,k-1} & T_{k,k-2} & T_{k,k-3} & \cdots \\ T_{k+1,k-1} & T_{k+1,k-2} & T_{k+1,k-3} & \cdots \\ T_{k+2,k-1} & T_{k+2,k-2} & T_{k+2,k-3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- Entries of Hankel Operator are taken from Toeplitz Operator

$$T_{ij} = \begin{cases} D_i & \text{for} \quad i = j \\ C_i A_{i-1} \dots A_{j+1} B_j & \text{for} \quad i < j \\ 0 & \text{for} \quad i > j \end{cases}$$

From Toeplitz to Hankel

- Toeplitz Operator — describes overall map from input to output
- Hankel Operator — describes map from past input to future output; is a part of Toeplitz Operator (incl. flip left-right)
- Construction for anti-causal Hankel operator follows same principle
- For LTI Systems: Toeplitz Operator has Toeplitz structure; Hankel Operator has Hankel structure
- What is Hankel operator good for? → watch next lecture … ☺