Time-Varying Systems and Computations

Unit 4.2

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QR Decomposition

- For a square, full rank matrix $$T = QR$$
- Orthonormal matrix $$Q^T Q = Q Q^T = 1$$
- Upper triangular matrix $$R = \begin{bmatrix} * & * & * & \cdots & * \\ & * & * & \cdots & * \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ & & & & * \end{bmatrix}$$
- Strategy – map T onto R by successive orthogonal elimination steps

Generalied Rotations

- Transformation goal

$$Rx = e_1 \qquad \|x\|^2 = \|y\|^2 = 1$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Properties of Rotation

$$R^T R = 1 \quad \det(R) = 1$$

- General transformation goal

$$Rx = y$$

$$R_x x = e_1 \qquad R_y y = e_1$$

$$R_y^T R_x = y$$

Parametrized Reflection

- Parameterized Generalized rotation for use in QR decomposition

$$R = \begin{bmatrix} x_1 & x_2^T \\ -x_2 & 1 - \frac{x_2 x_2^T}{1+x_1} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Exercise – check for properties of Generalized Rotation

Elimination Scheme

- Column by column – from the left

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} (R_{16}) \mapsto \begin{bmatrix} \star & \star & \star & \star \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{bmatrix} (R_{26}) \mapsto \begin{bmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{bmatrix} (R_{36}) \mapsto \begin{bmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & \cdot \end{bmatrix}$$

$$(R_{46}) \mapsto \begin{bmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} R \\ \hline 0 \end{bmatrix}$$

$$\underbrace{R_{46} R_{36} R_{26} R_{16}}_{Q^T} T = \begin{bmatrix} R \\ \hline 0 \end{bmatrix}$$

$$\det(Q) = 1$$

Computational Complexity

- For QR decomposition of m-by-n matrix

$$2n^2(m - \frac{n}{3}) \text{ floating point operations}$$

- Identical to Householder QR

Generalized Rotation QR

- Mostly unknown algorithm for computing QR decomposition

- Numerically robust and efficient (for sequential machines)

  hypthesis is that is numerically slightly better than Householder
- Same computational complexity than Householder

- Rotations are a group – final Q will be a rotation

- Some applications require Q to be a rotation (e.g. Computer Vision)