Time-Varying Systems and Computations

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Transfer Function LTV

Input/output map

$$y = T \cdot u$$

from state equation

$$\boldsymbol{Z}^{-1}\boldsymbol{x} = \boldsymbol{A} \cdot \boldsymbol{x} + \boldsymbol{B} \cdot \boldsymbol{u}$$

… extract state

$$\boldsymbol{x} = (\boldsymbol{1} - \boldsymbol{Z}\boldsymbol{A})^{-1}\boldsymbol{Z}\boldsymbol{B}\boldsymbol{u}$$

… plug in output

$$\boldsymbol{y} = \boldsymbol{C} \cdot \boldsymbol{x} + \boldsymbol{D} \cdot \boldsymbol{u}$$

Voilá …

Transfer function

$$T = D + C\left[1 - ZA\right]^{-1} ZB$$

What does this mean?

Toy Example

Finite dimensional, causal matrix

$$T = \begin{bmatrix} T_{11} & 0 & 0 & 0 & 0 \\ T_{21} & T_{22} & 0 & 0 & 0 \\ T_{31} & T_{32} & T_{33} & 0 & 0 \\ T_{41} & T_{42} & T_{43} & T_{44} & 0 \\ T_{51} & T_{52} & T_{53} & T_{54} & T_{55} \end{bmatrix} = D + C(1 - ZA)^{-1} ZB$$

$$A = \begin{bmatrix} \cdot & & & & \\ & A_2 & & & \\ & & A_3 & & \\ & & & A_4 & \\ & & & & \cdot \end{bmatrix} \quad B = \begin{bmatrix} B_1 & & & & \\ & B_2 & & & \\ & & B_3 & & \\ & & & B_4 & \\ & & & & \cdot \end{bmatrix} \quad C = \begin{bmatrix} \cdot & & & & \\ & C_2 & & & \\ & & C_3 & & \\ & & & C_4 & \\ & & & & C_5 \end{bmatrix} \quad D = \begin{bmatrix} D_1 & & & & \\ & D_2 & & & \\ & & D_3 & & \\ & & & D_4 & \\ & & & & D_5 \end{bmatrix}$$

Towards Transfer Function LTV

Neumann Expansion

$$
\begin{aligned}
(1 - ZA)^{-1} &= 1 + ZA + (ZA)^2 + (ZA)^3 + \dots \\
&= 1 + ZA + ZAZA + ZAZAZA + \dots
\end{aligned}
$$

Block Diagonal

$$
\mathbf{A} = \begin{bmatrix} \ddots & & \\ & \boxed{A_k} & \\ & & \ddots \end{bmatrix}
\qquad
[1 - ZA]^{-1} = \begin{bmatrix}
\ddots & & & & & \\
\ddots & 1 & & & & \\
\ddots & A_1 & 1 & & & \\
\ddots & A_2 A_1 & A_2 & 1 & & \\
\ddots & A_3 A_2 A_1 & A_3 A_2 & A_3 & 1 & \\
 & \vdots & \vdots & \vdots & & \ddots
\end{bmatrix}
$$

Components for Toy Example

$$[1 - ZA]^{-1} \left[\begin{array}{c|ccccc} 1 & & & & & \\ A_1 & 1 & & & & \\ A_2A_1 & A_2 & 1 & & & \\ A_3A_2A_1 & A_3A_2 & A_3 & 1 & & \\ A_4A_3A_2A_1 & A_4A_3A_2 & A_4A_3 & A_4 & 1 & \\ \hline A_5A_4A_3A_2A_1 & A_5A_4A_3A_2 & A_5A_4A_3 & A_5A_4 & A_5 & 1 \end{array}\right]$$

$$ZB = \left[\begin{array}{ccccc} 0 & & & & \\ B_1 & 0 & & & \\ & B_2 & 0 & & \\ & & B_3 & 0 & \\ & & & B_4 & 0 \\ \hline & & & & B_5 \end{array}\right]$$

Putting Stuff Together …

$$
= \left[\begin{array}{ccccc|c}
C_1 & & & & & \\
& C_2 & & & & \\
& & C_3 & & & \\
& & & C_4 & & \\
& & & & C_5 & \\
\hline
& & & & & C_6
\end{array}\right]
\left[\begin{array}{c|ccccc}
1 & & & & & \\
A_1 & 1 & & & & \\
A_2A_1 & A_2 & 1 & & & \\
A_3A_2A_1 & A_3A_2 & A_3 & 1 & & \\
A_4A_3A_2A_1 & A_4A_3A_2 & A_4A_3 & A_4 & 1 & \\
\hline
A_5A_4A_3A_2A_1 & A_5A_4A_3A_2 & A_5A_4A_3 & A_5A_4 & A_5 & 1
\end{array}\right]
\left[\begin{array}{ccccc}
0 & & & & \\
B_1 & 0 & & & \\
& B_2 & 0 & & \\
& & B_3 & 0 & \\
& & & B_4 & 0 \\
\hline
& & & & B_5
\end{array}\right]
$$

$$
= \left[\begin{array}{ccccc|c}
C_1 & & & & & \\
& C_2 & & & & \\
& & C_3 & & & \\
& & & C_4 & & \\
& & & & C_5 & \\
\hline
& & & & & C_6
\end{array}\right]
\left[\begin{array}{c|cccc}
0 & & & & \\
B_1 & 0 & & & \\
A_2B_1 & B_2 & 0 & & \\
A_3A_2B_1 & A_3B_2 & B_3 & 0 & \\
A_4A_3A_2B_1 & A_4A_3B_2 & A_4B_3 & B_4 & 0 \\
\hline
A_5A_4A_3A_2B_1 & A_5A_4A_3B_2 & A_5A_4B_3 & A_5B_4 & B_5
\end{array}\right]
$$

$$
= \left[\begin{array}{cccccc}
0 & & & & & \\
C_2B_1 & 0 & & & & \\
C_3A_2B_1 & C_3B_2 & 0 & & & \\
C_4A_3A_2B_1 & C_4A_3B_2 & C_4B_3 & 0 & & \\
C_5A_4A_3A_2B_1 & C_5A_4A_3B_2 & C_5A_4B_3 & C_5B_4 & 0 & \\
\hline
C_6A_5A_4A_3A_2B_1 & C_6A_5A_4A_3B_2 & C_6A_5A_4B_3 & C_6A_5B_4 & C_6B_5 & 0
\end{array}\right]
$$

Considering Empty Matrices

$$= \left[\begin{array}{cccccc} 0 & & & & & \\ C_2B_1 & 0 & & & & \\ C_3A_2B_1 & C_3B_2 & 0 & & & \\ C_4A_3A_2B_1 & C_4A_3B_2 & C_4B_3 & 0 & & \\ C_5A_4A_3A_2B_1 & C_5A_4A_3B_2 & C_5A_4B_3 & C_5B_4 & 0 & \\ \hline C_6A_5A_4A_3A_2B_1 & C_6A_5A_4A_3B_2 & C_6A_5A_4B_3 & C_6A_5B_4 & C_6B_5 & 0 \end{array}\right]$$

$$C_1 = [\cdot], \quad C_6 = [\cdot], \quad A_1 = [\cdot] \quad A_5 = [\cdot], \quad B_5 = [\cdot]$$

$$\begin{bmatrix} 0 & & & & \\ C_2B_1 & 0 & & & \\ C_3A_2B_1 & C_3B_2 & 0 & & \\ C_4A_3A_2B_1 & C_4A_3B_2 & C_4B_3 & 0 & \\ C_5A_4A_3A_2B_1 & C_5A_4A_3B_2 & C_5A_4B_3 & C_5B_4 & 0 \end{bmatrix}$$

Finally …

$$T = \begin{bmatrix} 0 & & & & \\ C_2B_1 & 0 & & & \\ C_3A_2B_1 & C_3B_2 & 0 & & \\ C_4A_3A_2B_1 & C_4A_3B_2 & C_4B_3 & 0 & \\ C_5A_4A_3A_2B_1 & C_5A_4A_3B_2 & C_5A_4B_3 & C_5B_4 & 0 \end{bmatrix} + \begin{bmatrix} D_1 & & & & \\ & D_2 & & & \\ & & D_3 & & \\ & & & D_4 & \\ & & & & D_5 \end{bmatrix}$$

Transfer Matrix T

$$T = \begin{bmatrix} D_1 & & & & \\ C_2B_1 & D_2 & & & \\ C_3A_2B_1 & C_3B_2 & D_3 & & \\ C_4A_3A_2B_1 & C_4A_3B_2 & C_4B_3 & D_4 & \\ C_5A_4A_3A_2B_1 & C_5A_4A_3B_2 & C_5A_4B_3 & C_5B_4 & D_5 \end{bmatrix}$$

Summary Example

Finite dimensional, causal matrix

$$T = \begin{bmatrix} T_{11} & 0 & 0 & 0 & 0 \\ T_{21} & T_{22} & 0 & 0 & 0 \\ T_{31} & T_{32} & T_{33} & 0 & 0 \\ T_{41} & T_{42} & T_{43} & T_{44} & 0 \\ T_{51} & T_{52} & T_{53} & T_{54} & T_{55} \end{bmatrix}$$

$$= D + C(1 - ZA)^{-1} ZB$$

$$= \begin{bmatrix} D_1 & 0 & 0 & 0 & 0 \\ C_2B_1 & D_2 & 0 & 0 & 0 \\ C_3A_2B_1 & C_3B_2 & D_3 & 0 & 0 \\ C_4A_3A_2B_1 & C_4A_3B_2 & C_4B_3 & D_4 & 0 \\ C_5A_4A_3A_2B_1 & C_5A_4A_3B_2 & C_5A_4B_3 & C_5B_4 & D_5 \end{bmatrix}$$

$$A = \begin{bmatrix} \cdot & & & & \\ & A_2 & & & \\ & & A_3 & & \\ & & & A_4 & \\ & & & & \cdot \end{bmatrix}$$

$$B = \begin{bmatrix} B_1 & & & & \\ & B_2 & & & \\ & & B_3 & & \\ & & & B_4 & \\ & & & & \cdot \end{bmatrix}$$

$$C = \begin{bmatrix} \cdot & & & & \\ & C_2 & & & \\ & & C_3 & & \\ & & & C_4 & \\ & & & & C_5 \end{bmatrix}$$

$$D = \begin{bmatrix} D_1 & & & & \\ & D_2 & & & \\ & & D_3 & & \\ & & & D_4 & \\ & & & & D_5 \end{bmatrix}$$

Empty Matrices

$$A = \begin{bmatrix} \cdot & & & & \\ & A_2 & & & \\ & & A_3 & & \\ & & & A_4 & \\ & & & & \cdot \end{bmatrix}$$

$$A_1 = [\cdot], \quad \dim A_1 = [0 \times 0]$$

$$A_5 = [\cdot], \quad \dim A_5 = [0 \times 0]$$

#	operation	result
1	$- * \vert$	$\cdot$
2	$\vert * -$	$[0]$
3	$\cdot * -$	$-$
4	$\vert * \cdot$	$\vert$
5	$\cdot * \vert$	illegal
6	$- * [a]$	$-$
7	$[a] * \vert$	$\vert$
8	$\begin{bmatrix} - & - \end{bmatrix} \begin{bmatrix} \vert \\ \vert \end{bmatrix}$	$\cdot$

Extend matrix multiplication

(Check with Matlab)

Transfer Function

- Compact representation of matrix T
- Representation builds on state-space parameters (A_k, B_k, C_k, D_k)
  in block diagonal form and Shift matrix Z
- Transfer Function T is <u>identical</u> to impulse response matrix T