Time-Varying Systems and Computations

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02.2 WS 2024

Time-Varying State-Space Equations

$$\begin{aligned} x_{k+1} &= A_k \cdot x_k + B_k \cdot u_k \\ y_k &= C_k \cdot x_k + D_k \cdot u_k \end{aligned}$$

$$\begin{bmatrix} x_{k+1} \\ y_k \end{bmatrix} = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] \cdot \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

Block-Diagonal Representation Pt.1

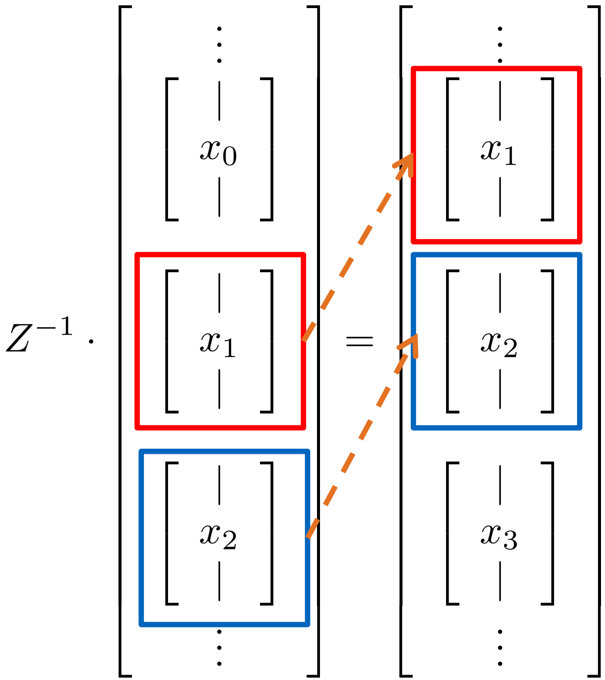
Handling a complete system → state-space equations in vectorized/block-diagonalized form

(State evolution equation)

$$
\begin{bmatrix} \vdots \\ \begin{bmatrix} | \\ x_1 \\ | \end{bmatrix} \\ \begin{bmatrix} | \\ x_2 \\ | \end{bmatrix} \\ \begin{bmatrix} | \\ x_3 \\ | \end{bmatrix} \\ \vdots \end{bmatrix}
=
\begin{bmatrix} \ddots & & & \\ & [A_0] & & \\ & & [A_1] & \\ & & & [A_2] \\ & & & & \ddots \end{bmatrix}
\cdot
\begin{bmatrix} \vdots \\ \begin{bmatrix} | \\ x_0 \\ | \end{bmatrix} \\ \begin{bmatrix} | \\ x_1 \\ | \end{bmatrix} \\ \begin{bmatrix} | \\ x_2 \\ | \end{bmatrix} \\ \vdots \end{bmatrix}
+
\begin{bmatrix} \ddots & & & \\ & [B_0] & & \\ & & [B_1] & \\ & & & [B_2] \\ & & & & \ddots \end{bmatrix}
\cdot
\begin{bmatrix} \vdots \\ [u_0] \\ [u_1] \\ [u_2] \\ \vdots \end{bmatrix}
$$

Block-Diagonal Representation Pt.2

Shifting up the state vector



Shift-up operator

$$Z^{-1} = \begin{bmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{bmatrix}$$

Shift-down operator

$$Z = \begin{bmatrix} \ddots & & & & & \\ \ddots & 0 & & & & \\ & 1 & 0 & & & \\ & & 1 & 0 & & \\ & & & 1 & \ddots & \\ & & & & \ddots & \ddots \end{bmatrix}$$

Block-Diagonal Representation Pt.3

Handling a complete system → output equation in vectorized/block-diagonalized form

(Output equation)

$$
\begin{bmatrix} \vdots \\ [y_1] \\ [y_2] \\ [y_3] \\ \vdots \end{bmatrix}
=
\begin{bmatrix} \ddots & & & & \\ & [C_0] & & & \\ & & [C_1] & & \\ & & & [C_2] & \\ & & & & \ddots \end{bmatrix}
\cdot
\begin{bmatrix} \vdots \\ \begin{bmatrix} | \\ x_0 \\ | \end{bmatrix} \\ \begin{bmatrix} | \\ x_1 \\ | \end{bmatrix} \\ \begin{bmatrix} | \\ x_2 \\ | \end{bmatrix} \\ \vdots \end{bmatrix}
+
\begin{bmatrix} \ddots & & & & \\ & [D_0] & & & \\ & & [D_1] & & \\ & & & [D_2] & \\ & & & & \ddots \end{bmatrix}
\cdot
\begin{bmatrix} \vdots \\ [u_0] \\ [u_1] \\ [u_2] \\ \vdots \end{bmatrix}
$$

State Space Equations – Block Diagonal

Putting it all together

$$Z^{-1}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{C}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$\mathbf{A} = \begin{bmatrix} \ddots & & \\ & \boxed{A_k} & \\ & & \ddots \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \ddots & & \\ & \boxed{B_k} & \\ & & \ddots \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \ddots & & \\ & \boxed{C_k} & \\ & & \ddots \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} \ddots & & \\ & \boxed{D_k} & \\ & & \ddots \end{bmatrix}$$

What can vary with time ?

Input Dimension	Output Dimension	State Dimension	Matrix Dimensions	Matrix Entries
m_k	n_k	d_k	$d_{k+1} \times d_k$	

$$\begin{bmatrix} \vdots \\ \begin{bmatrix} u_1 \end{bmatrix} \\ \begin{bmatrix} u_2 \end{bmatrix} \\ \begin{bmatrix} u_{31} \\ u_{32} \\ u_{33} \end{bmatrix} \\ \vdots \end{bmatrix} \quad \begin{bmatrix} \vdots \\ \begin{bmatrix} y_1 \end{bmatrix} \\ \begin{bmatrix} y_{21} \\ y_{22} \end{bmatrix} \\ \begin{bmatrix} y_3 \end{bmatrix} \\ \vdots \end{bmatrix} \quad \begin{bmatrix} \vdots \\ \begin{bmatrix} x_0 \end{bmatrix} \\ \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \\ \begin{bmatrix} x_2 \end{bmatrix} \\ \vdots \end{bmatrix}$$

$$A = \begin{bmatrix} \ddots & & & \\ & \begin{bmatrix} a_{11} \end{bmatrix}_0 & & \\ & & \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}_1 & \\ & & & \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}_2 \\ & & & & \ddots \end{bmatrix}$$

Final Remarks

LTV Systems ←→ LTI Systems

Block Diagonal Representation ←→ Frequency Domain Representation

(z-Transforms)

Time-varying systems theory allows to handle

- … General matrices beyond Toeplitz
- … Finite dimensional matrices
- … Systems with varying number of inputs/outputs
- … Systems with varying system dynamics