Time-Varying Systems and Computations

Klaus Diepold

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Time-Varying State-Space Equations

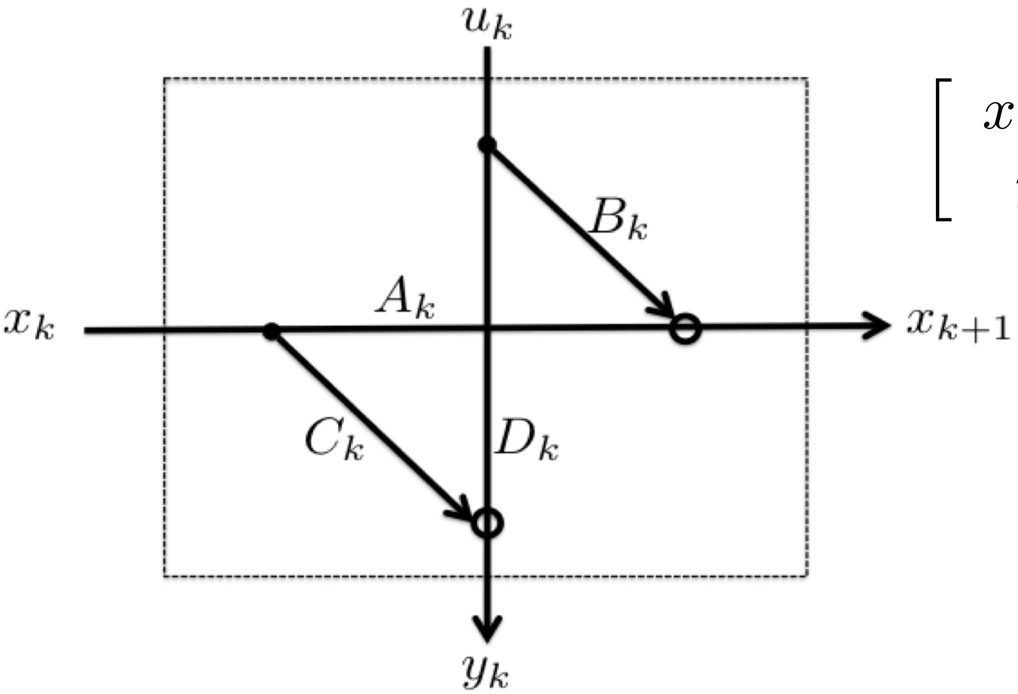
Basic State-Space Equations

$$\begin{aligned} x_{k+1} &= A_k \cdot x_k + B_k \cdot u_k \\ y_k &= C_k \cdot x_k + D_k \cdot u_k \end{aligned}$$

Block Notation

$$\begin{bmatrix} x_{k+1} \\ y_k \end{bmatrix} = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] \cdot \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

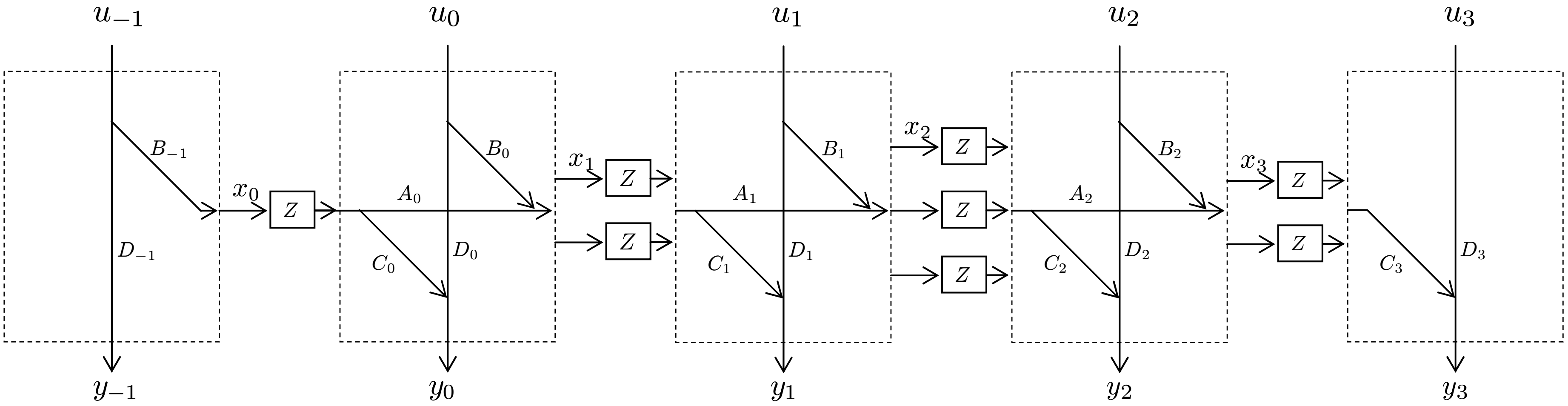
Elementary Block – Signal Flow Graph



$$\begin{bmatrix} x_{k+1} \\ y_k \end{bmatrix} = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] \cdot \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

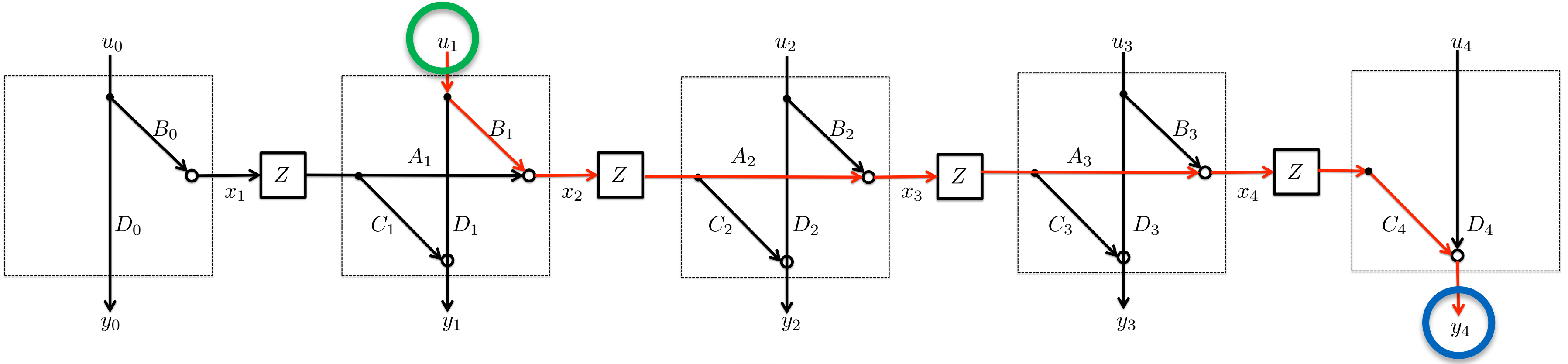
Time-Varying State-Space Model

Causal System Model → Composition of Elementary Building Blocks



State-Space Model - Impulse Response

Impulse
Input at time k=1



Output at time k=4

$$y_4 = C_4 A_3 A_2 B_1 \cdot u_1$$

Causal Impulse Response Matrix

Impuls at time k=1

$$
\boldsymbol{T} = \begin{bmatrix}
 & k=0 & k=1 & k=2 & \\
\ddots & \vdots & \vdots & \vdots & \\
\ddots & 0 & \vdots & \vdots & \\
\ddots & D_0 & 0 & \vdots & \\
\ddots & C_1B_0 & D_1 & 0 & \\
\ddots & C_2A_1B_0 & C_2B_1 & D_2 & \ddots \\
\ddots & C_3A_2A_1B_0 & C_3A_2B_1 & C_3B_2 & \ddots \\
 & C_4A_3A_2A_1B_0 & C_4A_3A_2B_1 & C_4A_3B_2 & \ddots \\
 & \vdots & C_5A_4A_3A_2B_1 & C_5A_4A_3B_2 & \ddots \\
 & \vdots & \vdots & C_6A_5A_4A_3B_2 & \ddots \\
 & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
$$

Response at time k=4

Causal Impulse Response Matrix LTV

Time-Variant System → no Toeplitz matrix anymore

$$k = 0 \qquad\qquad k = 1 \qquad\qquad k = 2$$

$$\boldsymbol{T} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \ddots & 0 & \vdots & \vdots & \\ \ddots & D_0 & 0 & \vdots & \\ \ddots & C_1B_0 & D_1 & 0 & \\ \ddots & C_2A_1B_0 & C_2B_1 & D_2 & \ddots \\ \ddots & C_3A_2A_1B_0 & C_3A_2B_1 & C_3B_2 & \ddots \\ & C_4A_3A_2A_1B_0 & C_4A_3A_2B_1 & C_4A_3B_2 & \ddots \\ & \vdots & C_5A_4A_3A_2B_1 & C_5A_4A_3B_2 & \ddots \\ & \vdots & \vdots & C_6A_5A_4A_3B_2 & \ddots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Values change along diagonals

Causal Impulse Response Matrix LTI

All A_k, B_k, C_k, D_k are identical, i.e. we have A, B, C, D for all k

→ Impulse response is back to infinite dimensional Toeplitz matrix

$$\mathbf{A} = \begin{bmatrix} \ddots & & & & \\ & A & & & \\ & & \boxed{A} & & \\ & & & A & \\ & & & & \ddots \end{bmatrix}$$

$$\vdots$$

$$\mathbf{D} = \begin{bmatrix} \ddots & & & & \\ & D & & & \\ & & \boxed{D} & & \\ & & & D & \\ & & & & \ddots \end{bmatrix}$$

$$T = \begin{bmatrix} \ddots & \vdots & & & & \\ \ddots & D & & & & \\ \ddots & CB & D & & & \\ \ddots & CAB & CB & D & & \\ \ddots & CA^2B & CAB & CB & D & \\ \ddots & CA^3B & CA^2B & CAB & CB & \ddots \\ & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Finite Dimensional Matrices

LTV Theory → covers finite dimensional matrices with/without Toeplitz structure

$$T = \begin{bmatrix} [\cdot] & & & & \\ & [\cdot] & & & \\ & & \boxed{\mathrm{T}} & & \\ & & & [\cdot] & \\ & & & & [\cdot] \end{bmatrix} = \begin{bmatrix} [\cdot] & & & & \\ & [\cdot] & & & \\ & & \begin{bmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \vdots & & & \vdots \\ T_{m1} & T_{m2} & \dots & T_{mn} \end{bmatrix} & & \\ & & & [\cdot] & \\ & & & & [\cdot] \end{bmatrix}$$

→ Matrix T is bordered by empty matrices

→ What is the theory good for?

Example: Lower Triangular Matrix

Columns of matrix T contain the time-varying impulse response

$$T = \begin{bmatrix} 1 & & & \\ 1/2 & 1 & & \\ 1/6 & 1/3 & 1 & \\ 1/24 & 1/12 & 1/4 & 1 \end{bmatrix}$$

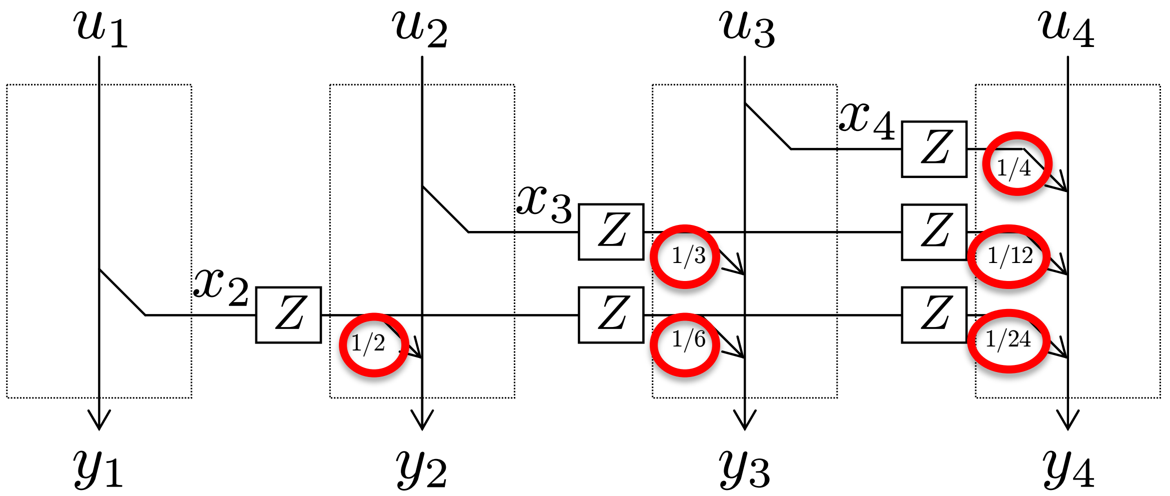
$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

Compute

$$T \cdot u = y$$

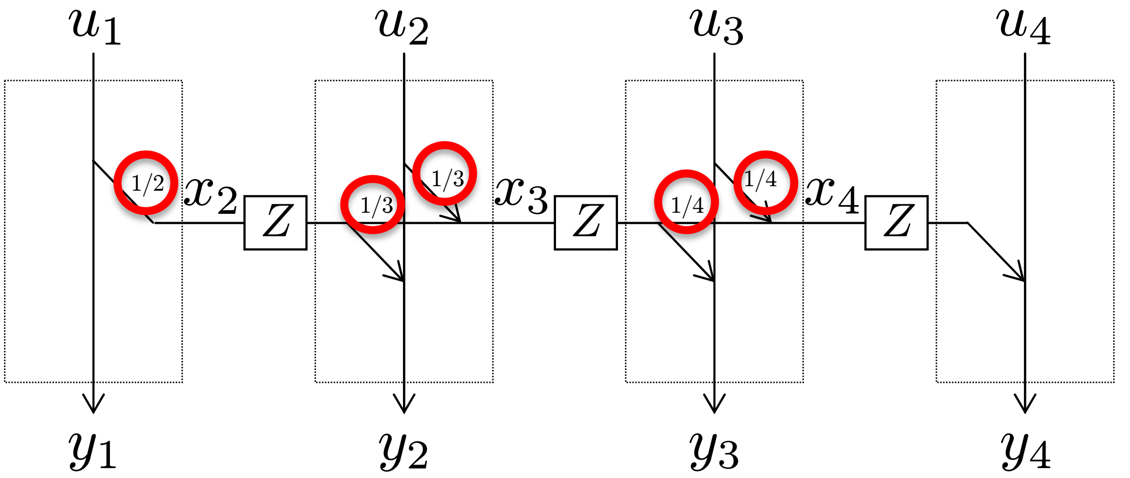
Direct State-Space Model for Matrix T



$$T = \begin{bmatrix} 1 & & & \\ 1/2 & 1 & & \\ 1/6 & 1/3 & 1 & \\ 1/24 & 1/12 & 1/4 & 1 \end{bmatrix}$$

6 Multiplications
5 Additions
6 Latches

Simplified State-Space Model for T



$$T = \begin{bmatrix} 1 & & & \\ 1/2 & 1 & & \\ 1/6 & 1/3 & 1 & \\ 1/24 & 1/12 & 1/4 & 1 \end{bmatrix}$$

5 Multiplications
4 Additions
3 Latches

Computational Savings …

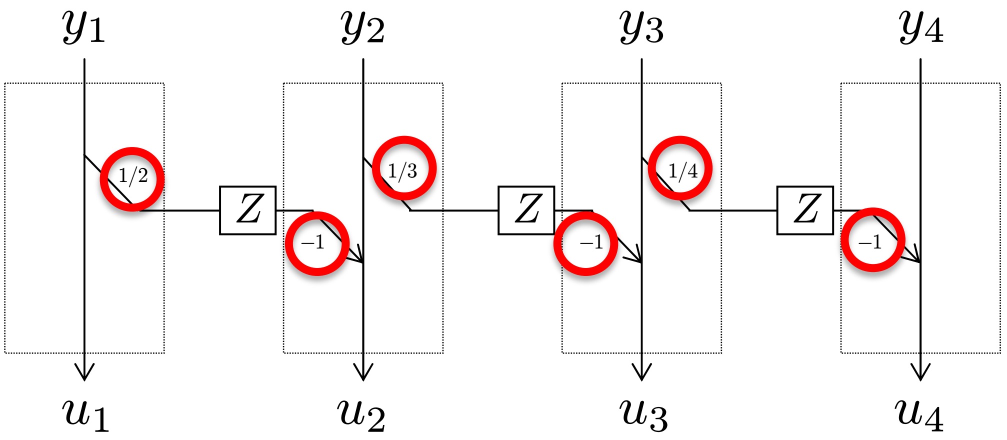
- for our toy example

6 Multiplications	→	5 Multiplications	- 17 %	
5 Additions		4 Additions	- 20 %	Reduction
6 Latches		3 Latches	- 50 %	

- … appear to be moderate … for a super simple example …
- Much more impressive savings for large scale tasks

- → Any underlying structure – invisible to the eye?

State-Space Model for the Inverse Matrix



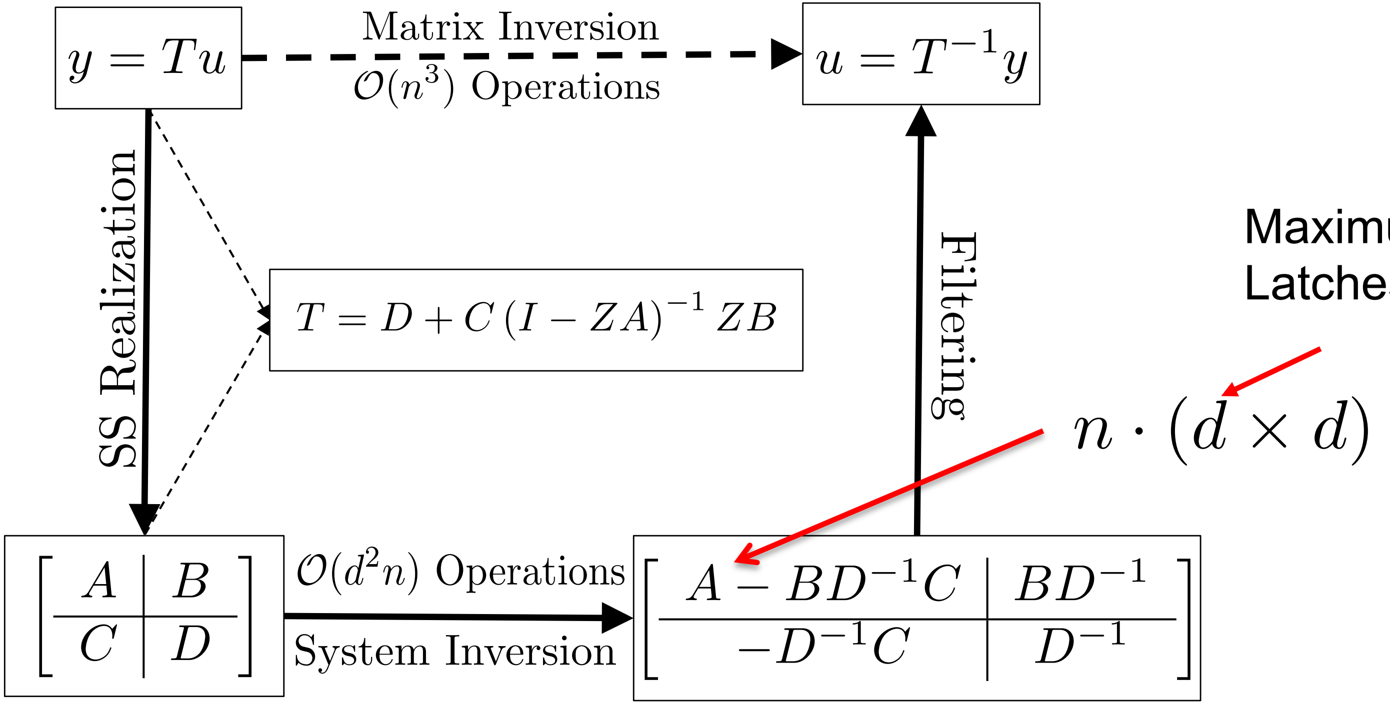
$$
\boldsymbol{T}^{-1} = \begin{bmatrix} 1 & & \mathbf{0} & \\ -1/2 & 1 & & \\ 0 & -1/3 & 1 & \\ 0 & 0 & -1/4 & 1 \end{bmatrix}
$$

Bi-diagonal Matrix
→ strong (invisible) structure

Questions

- How do we find a state-space representation for a given matrix T
  - How do we determine the values/dimensions of A_k, B_k, C_k, D_k ?
- How can we identify if the matrix T exhibits an exploitable structure?
  - When can we get a low-complexity SS model for a matrix ?
- Can we estimate the complexity of SS-Model for a given matrix T ?
- Can we approximate a given matrix T with a matrix T‘ that has a low complexity state-space model ?

Matrix Inversion via State-Space Model



Maximum number of Latches per state