Lecture 1

Time-variant Systems and Quasi-separable systems: an overview

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Overview

- Dynamical System Theory as a discipline
- The basic notions
- System identification and realization
- The main system operations
- What is it all good for?
- The connection between systems and matrices
- Numerics
- Envoy

What is a discipline?

- a consistent body of theory and practice
- characterized by some key notions
- having a specific methodology and main results
- created by some "patriarchs"

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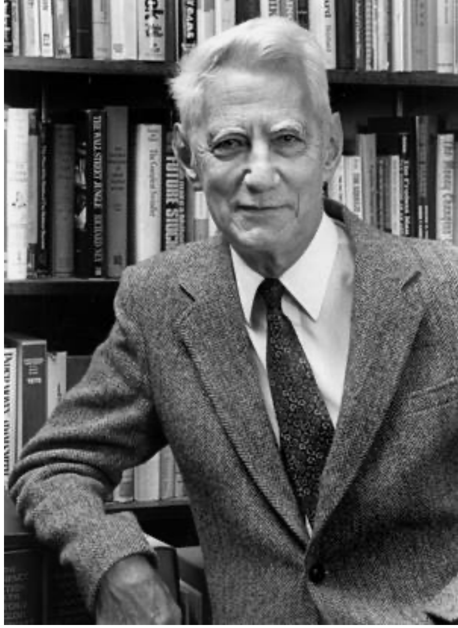

Examples

- Signal Processing [Fourier, Wiener]
- Control [Bode, Nyquist]
- Information Theory [Shannon]
- Electromagnetism [Maxwell, Herz]
- Solid State Physics [Schottky, Shockley]
- Electronics [Noyce, Kilby]
- Computer Science [von Neumann]

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etc...

Dynamical System Theory: a definition?

The theory that describes the evolution of a system as time progresses

Key notion #1: the STATE of the system: "what the system remembers from its past"

Key notion #2: the EVOLUTION of the state (i.e. the dynamics)

Key notion #3: the BEHAVIOR of the system (i.e. how the system looks from the outside)

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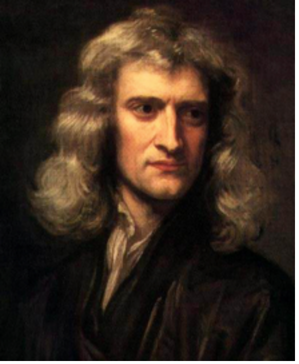

A bit of history

- the roots are definitely in Mechanics: Newton recognized that the state of a mechanical system consists of *positions* **and** *velocities*
- a new impetus came when Kalman recognized that to control a system, one needs knowledge of its state: *the state had to be estimated*
- this lead to dynamical system theory as a new discipline in applied mathematics

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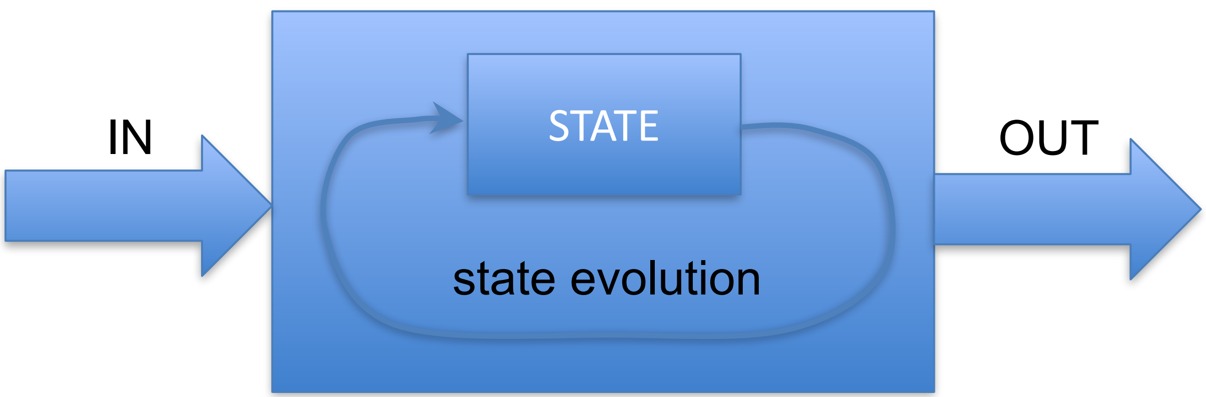
The approach here

We shall approach the topic from an input-output point of view – there exists a "purer" approach, in which an input-output map is not assumed (Willems: behavioral system theory).

Static view:



Behavior: the resulting map from inputs to outputs

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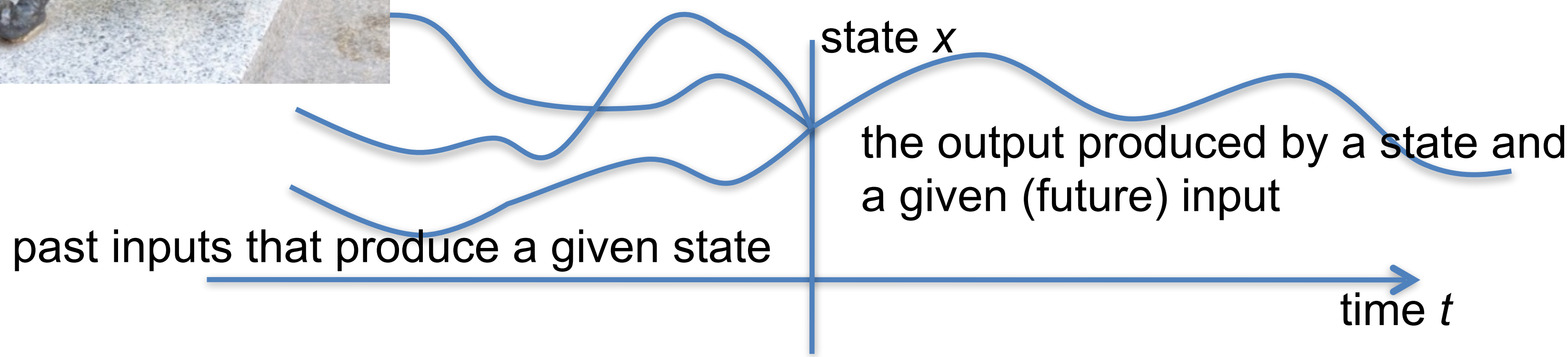
The dynamic view



In equation: continuous time:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t) \\ y(t) = g(x(t), u(t), t) \end{cases}$$

discrete time:

$$\begin{cases} x(k+1) = f_k(x(k), u(k)) \\ y(k) = g_k(x(k), u(k)) \end{cases}$$

practical consideration: there is no harm to replace continuous time by discrete time – it makes the discussions much easier!

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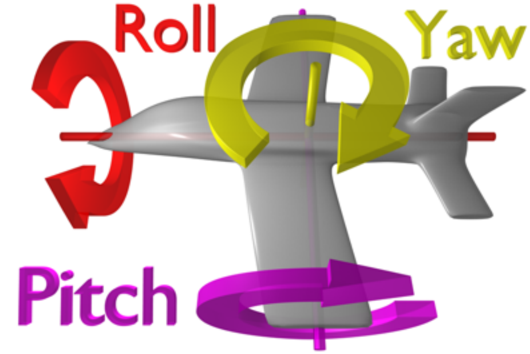
Basic notions

- the STATE: a time dependent vector
- the EVOLUTION of the state: a difference equation
- REACHABILITY: how a state can be reached by past inputs (important for control)
- OBSERVABILITY: how one can estimate the state of a system by observing it (important for estimation)
- MINIMALITY: no superfluous states!

the state?



- Mechanical system: *position and velocity*
- Computer: *relevant memory (data storage, switches)*
- Automaton: *control states, routing states*
- Airplane: *position, velocity, roll, yaw and pitch, angles and velocities*
- Process plant: *pressure, temperature, concentrations*

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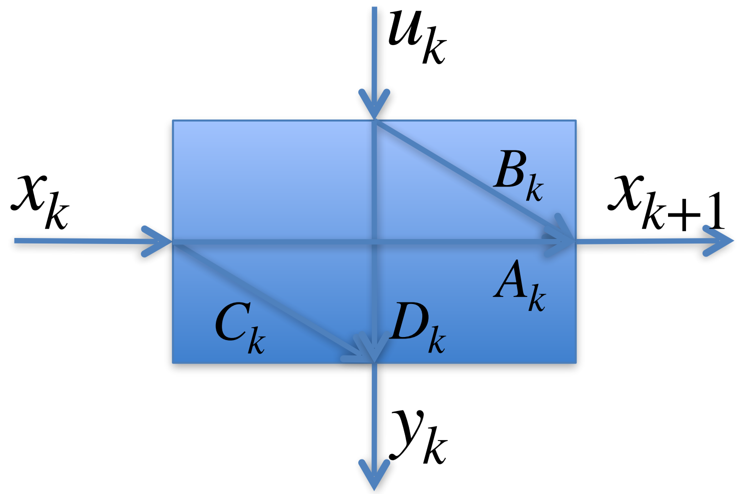
the analysis is easier for linear systems...

$$\begin{cases} x_{k+1} = A_k x_k + B_k u_k \\ y_k = C_k x_k + D_k u_k \end{cases}$$

("causal" system)



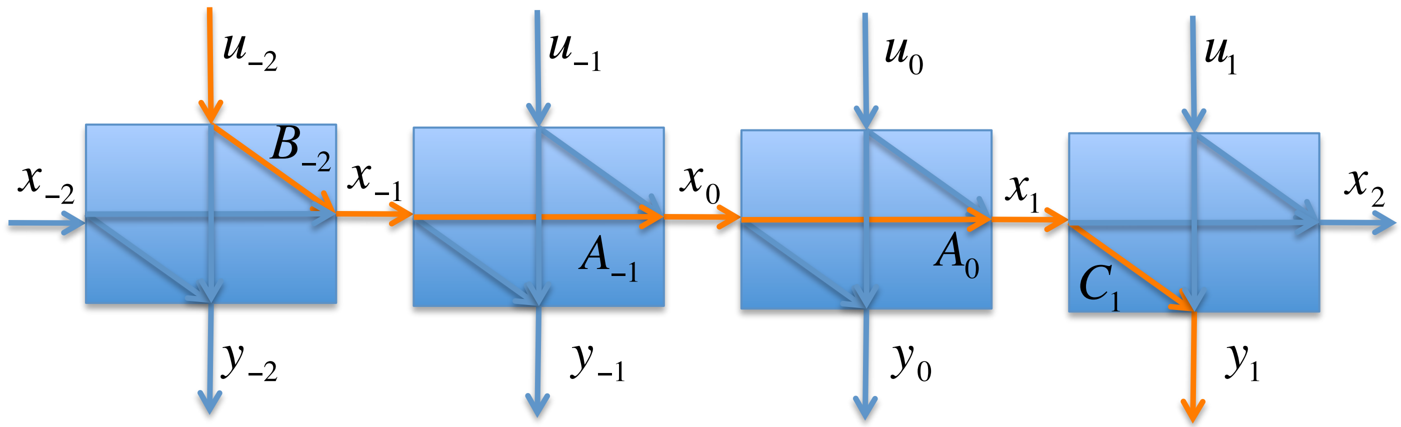
$$\begin{bmatrix} x_{k+1} \\ y_k \end{bmatrix} = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

REACHABILITY: can any state (at each time point) be reached by some past inputs?

OBSERVABILITY: is every state characterized by the resulting response?

let's develop the algebra...

the input-output operator (causal)



$$y = Tu$$

$$\begin{bmatrix} \vdots \\ y_{-2} \\ y_{-1} \\ \boxed{y_0} \\ y_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & & & & \\ & \ddots & & & 0 & \\ & & \ddots & & & \\ & & & D_0 & & \\ & C_1 A_0 A_{-1} B_{-2} & & & \ddots & \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ u_{-2} \\ u_{-1} \\ \boxed{u_0} \\ u_1 \\ \vdots \end{bmatrix}$$

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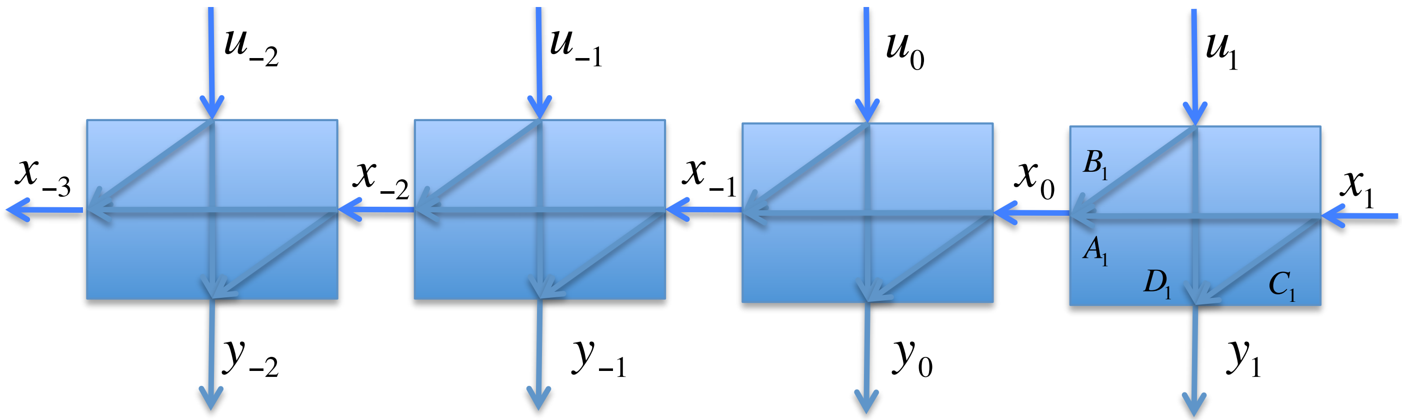
input-output anti-causal



$$T = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \cdot^{\cdot^{\cdot}} \\ & D_{-1} & C_{-1}B_0 & C_{-1}A_0B_1 & \ddots \\ & & \boxed{D_0} & C_0B_1 & \ddots \\ & \mathbf{0} & & D_1 & \ddots \\ & & & & \ddots \end{bmatrix}$$

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Representations

Linear Time-Invariant:
$$\begin{cases} U(z) = \cdots + u_{-1}z^{-1} + u_0 + u_1 z + \dots \\ Y(z) = \cdots + y_{-1}z^{-1} + y_0 + y_1 z + \cdots \\ \quad T(z) = D + C(I - zA)^{-1} zB \end{cases}$$

Time-variant: define *block diagonal operators*

instantaneous:

$$A = \begin{bmatrix} \ddots & & & \\ & A_{-1} & & & \\ & & \boxed{A_0} & & \\ & & & A_1 & \\ & & & & \ddots \end{bmatrix}, B = \begin{bmatrix} \ddots & & & \\ & B_{-1} & & & \\ & & \boxed{B_0} & & \\ & & & B_1 & \\ & & & & \ddots \end{bmatrix}$$ etc...

shifts, causal:

$$Z = \begin{bmatrix} \ddots & & & \\ \ddots & 0 & & \\ & I & \boxed{0} & \\ & & I & 0 & \\ & & & \ddots & \ddots \end{bmatrix}$$

anti-causal:

$$Z' = \begin{bmatrix} \ddots & \ddots & & \\ & 0 & I & \\ & & 0 & I \\ & & & \ddots & \ddots \end{bmatrix}$$

Resulting transfer operators:

$$T = D + C(I - ZA)^{-1} ZB \qquad T = D + C(I - Z'A)^{-1} Z'B$$

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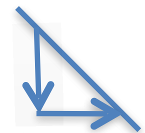
Stability?

The state evolves as:

$$(I - ZA)^{-1} = I + ZA + ZAZA + \cdots$$

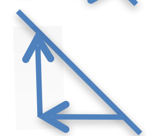
Define diagonal shifts:

$$A^{\langle +1 \rangle} = ZAZ'$$



(forward)

$$A^{\langle -1 \rangle} = Z'AZ$$



(backward)

$$(I - ZA)^{-1} = I + ZA + Z^2 A^{\langle -1 \rangle} A + Z^3 A^{\langle -2 \rangle} A^{\langle -1 \rangle} A + \cdots$$

continuous product should decrease exponentially
(called "u.e.s."= uniform exponentially stable)

example of unstable:

$$\begin{bmatrix} 1 & & & & \\ -2 & 1 & & & \\ & -2 & 1 & & \\ & & -2 & 1 & \\ & & & -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 4 & 2 & 1 & & \\ 8 & 4 & 2 & 1 & \\ 16 & 8 & 4 & 2 & 1 \end{bmatrix}$$

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reconciling matrices and systems

Linear time-invariant systems have doubly infinite Toeplitz input-output operators:

$$T = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \iddots \\ \ddots & T_0 & T_{-1} & T_{-2} & \ddots \\ \ddots & T_1 & \boxed{T_0} & T_{-1} & \ddots \\ \ddots & T_2 & T_1 & T_0 & \ddots \\ \iddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

zero dimensions: a necessary extension of matrix theory

One row, no column: $[||]$
One column, no row: $[-]$
No row, no column: $[\cdot]$
Multiplication rules: $[||][-] = [0]; \; [-][||] = [\cdot]$ (dimensions must match)
Unit of zero dimension: $[\cdot]$

(why? because $[\,\cdot\,][\,\cdot\,] = [\,\cdot\,]$!)

Note: extends to more columns and row, often abbreviated, as in $[\,-\,-\,-\,] \sim [\,-\,]$!

embedding: $$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \bullet & - & \bullet & \cdots \\ \cdots & | & T & | & \cdots \\ \cdots & \bullet & - & \bullet & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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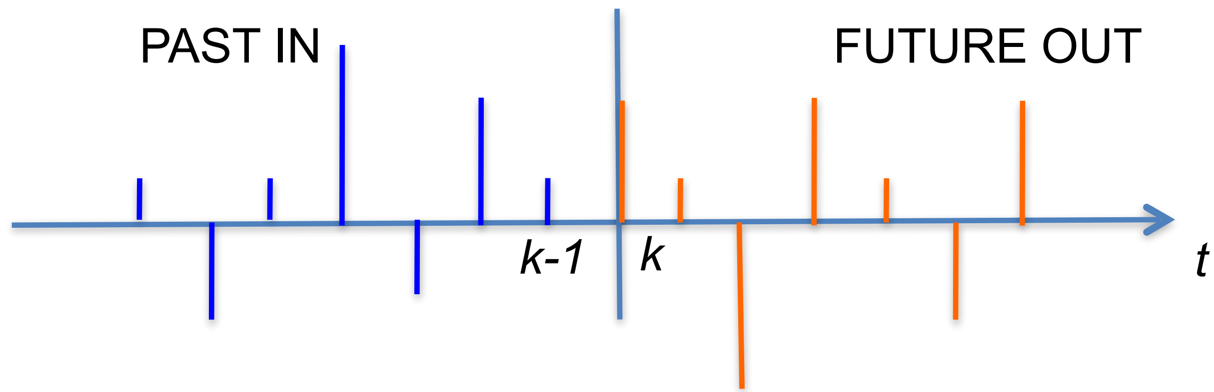
Past-present-future: the Hankel operator

the causal case: H_k maps past inputs up to *k-1* to future outputs from *k* on:



(present is part of future!)

past in

$$\begin{bmatrix} & & \ddots & & \\ \cdots & C_kA_{k-1}B_{k-2} & C_kB_{k-1} & D_k & \\ \cdots & C_{k+1}A_kA_{k-1}B_{k-2} & C_{k+1}A_kB_{k-1} & & \ddots \\ \cdots & C_{k+2}A_{k+1}A_kA_{k-1}B_{k-2} & C_{k+2}A_{k+1}A_kB_{k-1} & & \\ \iddots & \vdots & \vdots & & \end{bmatrix}$$

future out

$$H_k = \begin{bmatrix} T_{k,k-1} & T_{k,k-2} & \cdots \\ T_{k+1,k-1} & T_{k+1,k-2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

System identification: factoring the Hankel operator

Given T, what is a minimal realization $\{A, B, C, D\}$?

The answer: it is given by a minimal factorization of **each** Hankel operator H_{Tk}:

(generalized Kronecker theorem)

$$H_{\mathrm{Tk}} = \begin{bmatrix} C_k \\ C_{k+1}A_k \\ C_{k+2}A_{k+1}A_k \\ \vdots \end{bmatrix} \begin{bmatrix} \cdots & A_{k-1}A_{k-2}B_{k-3} & A_{k-1}B_{k-2} & B_{k-1} \end{bmatrix}$$

observability operator $\mathbf{O}_k$ at k

reachability operator $\mathbf{R}_k$ at

state

maps to state

maps to output

minimal factorization $\equiv$ choosing complementary bases

Identification (2) and normal forms

Remark: $\mathbf{O}_k = \begin{bmatrix} C_k \\ \mathbf{O}_{k+1} A_k \end{bmatrix}$, $\mathbf{R}_k = \begin{bmatrix} A_{k-1}\mathbf{R}_{k-1} & B_{k-1} \end{bmatrix}$

hence: $C_k = [\mathbf{O}_k]_k,\ B_k = [\mathbf{R}_{k+1}]_k,\ A_k = \mathbf{O}_{k+1}^{+}[\mathbf{O}_k]_{k+1:\infty}$

Input normal form: choose an orthonormal basis for all R_k

then $\begin{bmatrix} A_k & B_k \end{bmatrix}$ is co-isometric: $A_k A_k' + B_k B_k' = I$

Output normal form: choose an orthonormal basis for all O_k

then $\begin{bmatrix} A_k \\ C_k \end{bmatrix}$ will be isometric: $A_k' A_k + C_k' C_k = I$

Change of basis (causal case):

$$x_k = R_k \hat{x}_k \Rightarrow \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \mapsto \begin{bmatrix} R_{k+1}^{-1} A_k R_k & R_{k+1}^{-1} B_k \\ C_k R_k & D_k \end{bmatrix}$$

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Normalized from any minimal?

Solve a recursive Lyapunov-Stein equation:
e.g. to obtain the Output Normal Form

$$R_{k+1}R'_{k+1} = A_k R_k R'_k A'_k + B_k B'_k$$

(forward recursion)

Best method: R-Q factorization (square root algorithm):

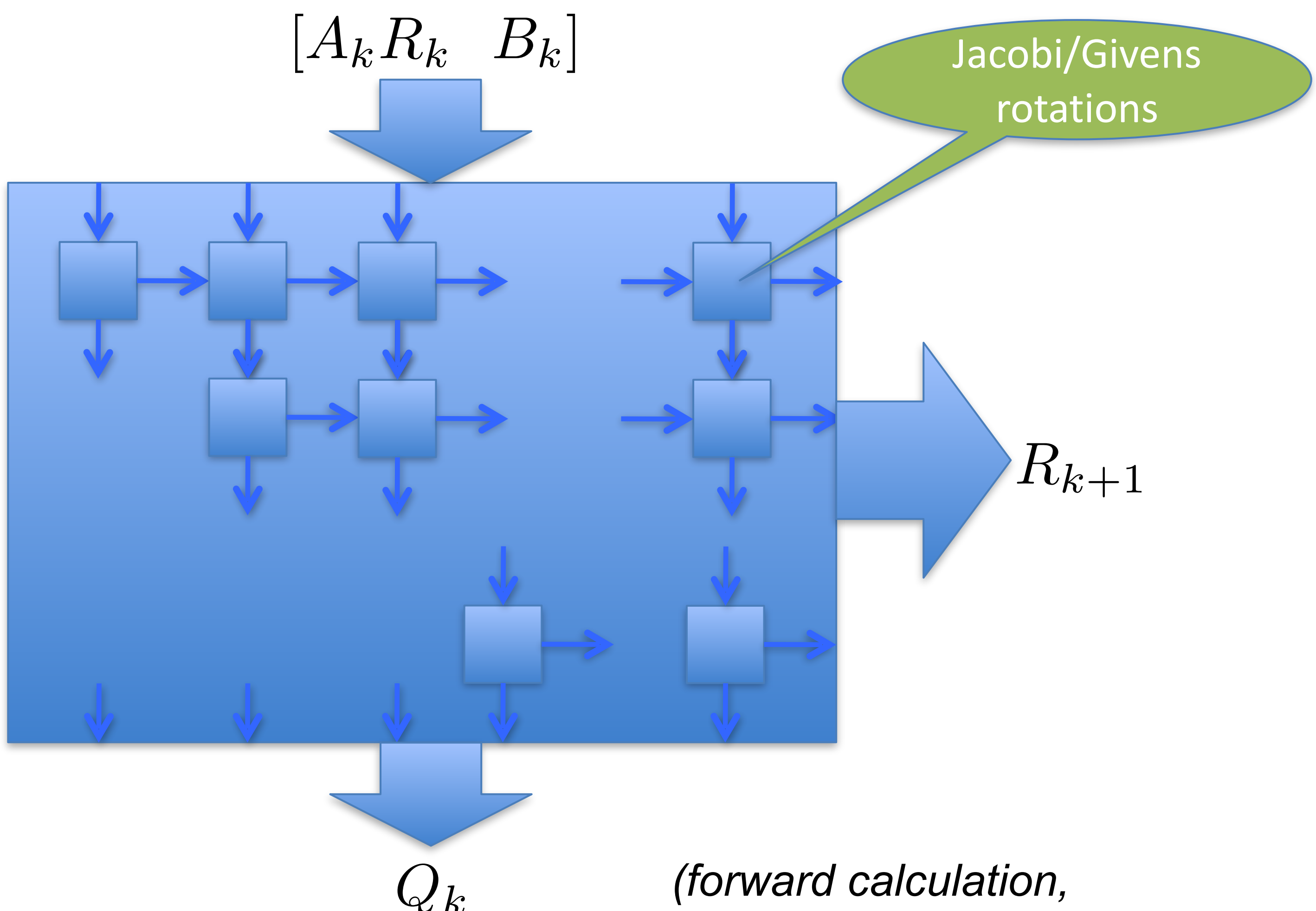
$$[A_k R_k \quad B_k] = [0 \quad R_{k+1}]Q_k$$

Unitary matrix

Compress columns

example: $\begin{bmatrix} \sqrt{2} & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/2 & -1/2 \\ -1/\sqrt{2} & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Use a parallel processor: (Gentleman-Kung array)



(forward calculation, stable when the system is stable)

What are the main system operations?

- System inversion!
  - signal/system estimation (Kalman filter e.g.)
  - stable system control
- System approximation!
  - system simulation
  - model order reduction
- System's eigenmodi!
    - *(needs system inversion)*

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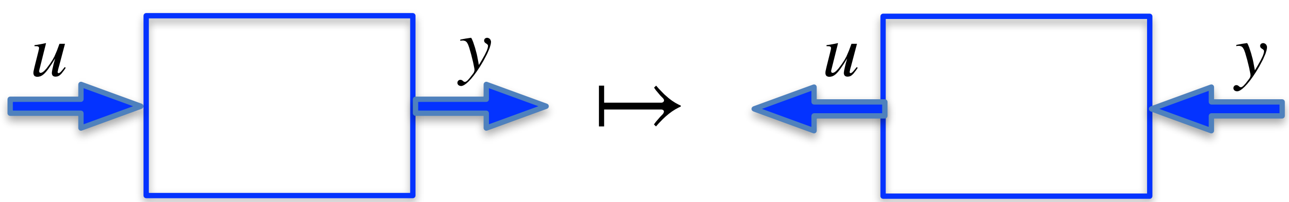
the main issue is...

how to invert a system in a stable way
very much related to the inversion of systems of equations: we'll discover some new methods based on reachability and observability!



$$C_c(I - ZA_c)^{-1}ZB_c + D + C_a(I - Z'A_a)^{-1}Z'B_a \quad \longmapsto \quad \text{inverse? or Moore-Penrose inverse?}$$

with additional properties:

- low quasi-separable complexity
- numerically stable operations (orthogonal transformations)

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warm up examples

half infinite systems:

$$\begin{bmatrix} \boxed{1} & & & \\ -1/2 & 1 & & \\ & -1/2 & 1 & \\ & & \ddots & \ddots \end{bmatrix}^{-1} = \text{‘}$$

$$\begin{bmatrix} \boxed{1} & & & \\ -2 & 1 & & \\ & -2 & 1 & \\ & & \ddots & \ddots \end{bmatrix}^{-1} = ?$$

examples (cnt'd)

easy to check:

$$\begin{bmatrix} \boxed{1} & & & \\ -1/2 & 1 & & \\ & -1/2 & 1 & \\ & & \ddots & \ddots \end{bmatrix}^{-1} = \begin{bmatrix} \boxed{1} & & & \\ 1/2 & 1 & & \\ 1/4 & 1/2 & 1 & \\ \iddots & \ddots & \ddots & \ddots \end{bmatrix}$$

but

$$\begin{bmatrix} \boxed{1} & & & \\ -2 & 1 & & \\ & -2 & 1 & \\ & & \ddots & \ddots \end{bmatrix}^{-1} ? = ? \begin{bmatrix} \boxed{1} & & & \\ 2 & 1 & & \\ 4 & 2 & 1 & \\ \iddots & \ddots & & \ddots \end{bmatrix}$$

unstable!

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but remark...

$$
\begin{bmatrix}
\ddots & & & & \\
\ddots & 1 & & & \\
 & -2 & \boxed{1} & & \\
 & & -2 & 1 & \\
 & & & \ddots & \ddots
\end{bmatrix}^{-1}
=
\begin{bmatrix}
\ddots & \ddots & \ddots & \ddots & \iddots \\
 & 0 & -1/2 & -1/4 & \ddots \\
 & & \boxed{0} & -1/2 & \ddots \\
 & & & 0 & \ddots \\
 & & & & \ddots
\end{bmatrix}
$$

(Toeplitz case...)

so what?

example (cnt'd)

$$\begin{bmatrix} \boxed{1} & & & \\ -2 & 1 & & \\ & -2 & 1 & \\ & & \ddots & \ddots \end{bmatrix}$$

is not invertible! Co-kernel: $\begin{bmatrix} \boxed{1} & 1/2 & 1/4 & \cdots \end{bmatrix}$

It does have a left inverse: $\begin{bmatrix} \boxed{0} & -1/2 & -1/4 & \cdots \\ & 0 & -1/2 & \ddots \\ & & 0 & \ddots \\ & & & \ddots \end{bmatrix}$

what is its right Moore-Penrose inverse?

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the solution?

To solve these problems, we need a new type of factorization:

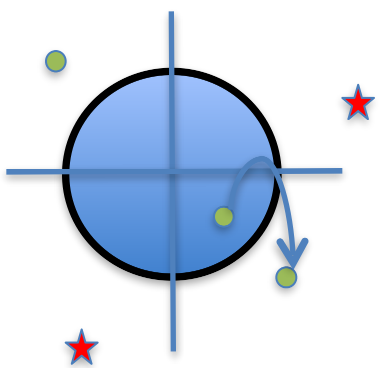
outer-inner factorization

Causal case: $$T = RU$$

Causally left invertible

Causally co-isometric

Toeplitz case: move zeros to outside the unit disc!



$$T^{\dagger} = U'R^{\dagger}$$

anti-causal

causal

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how to compute *U* and *R* from *T*?

the "square-root algorithm" again... let $T \sim_c \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

then $U \sim_c \begin{bmatrix} A_U & B_U \\ C_U & D_U \end{bmatrix}$ and $R \sim_c \begin{bmatrix} A & B_o \\ C & D_o \end{bmatrix}$ can be computed by a

(forward) recursive R-Q factorization: $\begin{bmatrix} AY & C \\ BY & D \end{bmatrix} = \begin{bmatrix} 0 & Y^{<-1>} & B_o \\ 0 & 0 & D_o \end{bmatrix} Q$

in which *Y* is a recursive intermediate diagonal with left inverse,
Q decomposes as

$$Q = \begin{bmatrix} C_n & D_n \\ A_U & B_U \\ C_U & D_U \end{bmatrix}$$

D_o also has a left inverse, and the causal kernel of *T* is given by

$$D_n + C_n (I - ZA_U)^{-1} ZB_U$$

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remarks on the sq. roots algorithm

- numerically stable *(only orthogonal transformations)*
- the intermediate Y is actually $Y = \mathbf{R}_A \mathbf{R}'_U$ *(i.e. the cross-correlation between the reachabilities of A and U).*
- the algorithm goes forward, needs starting data *(empty for finite matrices)*
- the outer factor may not be minimal.

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Which problems does this solve?

- the Kalman filter
- spectral factorization, the Wiener filter
- LU-factorization
- optimal control
- Moore-Penrose inversion of matrices
- the computation of kernels and ranges
- low complexity system inversion

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to be covered in the next lectures...

Numerical issues

- numerical conditioning of a problem: *a system is badly conditioned, when small perturbations in the input data causes large perturbations in the output (assuming infinite precision).*
- forward stability of an algorithm: *an algorithm is forward stable when numerical errors in the computation cause only small errors in the result.*
- backward stability of an algorithm: *an algorithm is backward stable, when numerical errors in the computation can be corrected by small variations in the input data (and exact mapping to the result).*

many physical problems are inherently ill-conditioned – that is the essence of scientific discovery! – a backward stable algorithm will allow good results *when sufficient numerical accuracy is used.*

The square-root algorithm is backward stable!

Traditionally, the problems detailed so far are solved via a Riccati equation: this is intrinsically numerically unstable.

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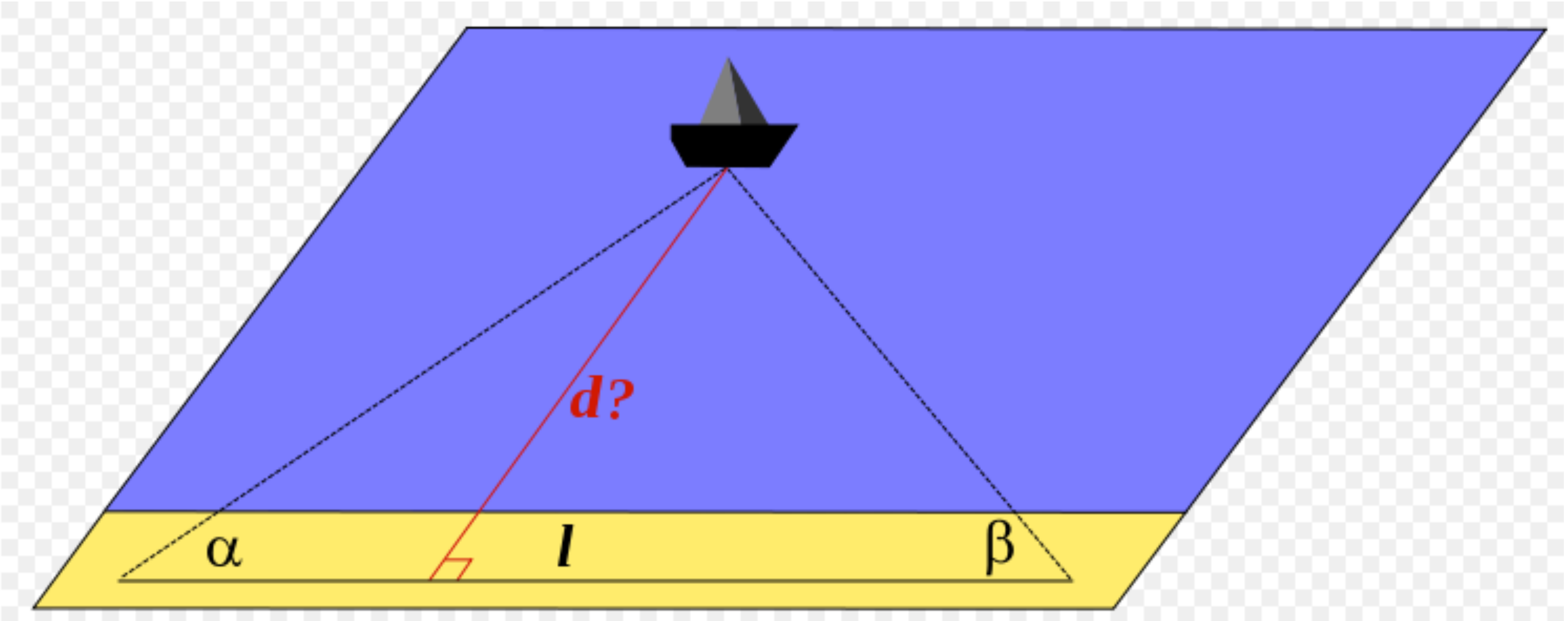
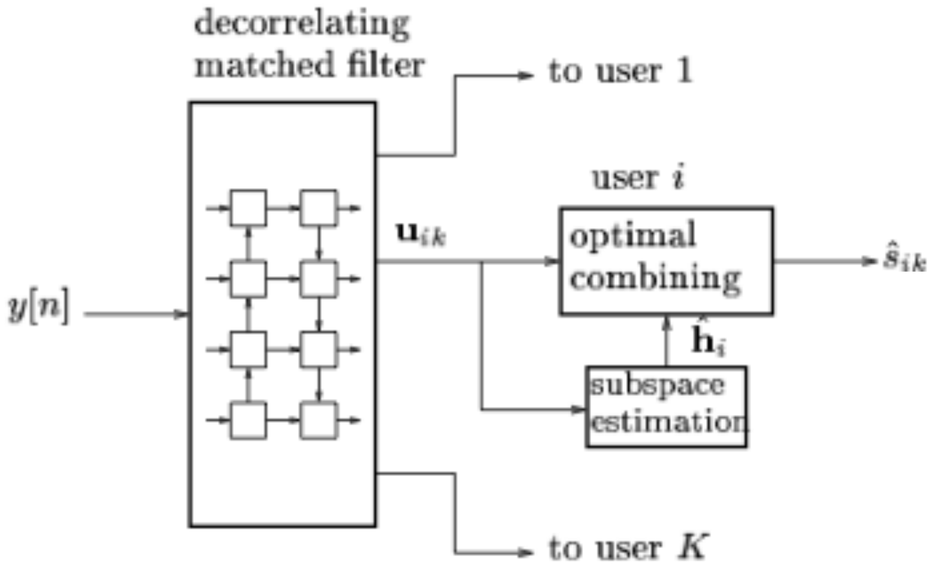
The simplest possible example of ill-conditioning?



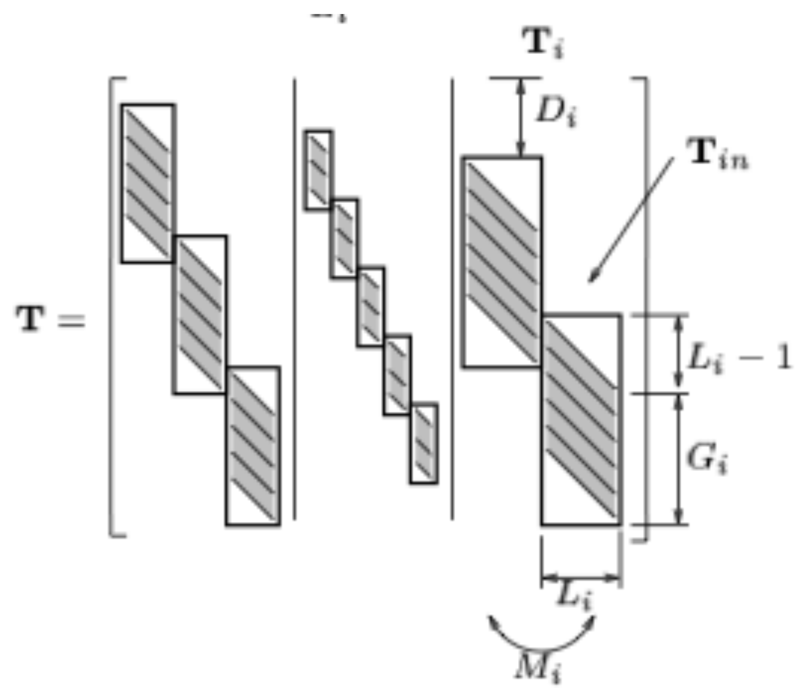
$$d = \frac{l \sin\alpha \sin\beta}{\sin(\alpha+\beta)} \approx \frac{l}{0}$$

A concrete telecom example of a typical quasi-separable case

The blind decorrelating Rake receiver for long-word WCDMA (Tong, vdVeen, PD)

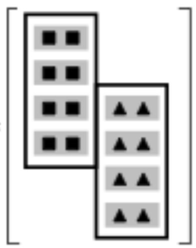


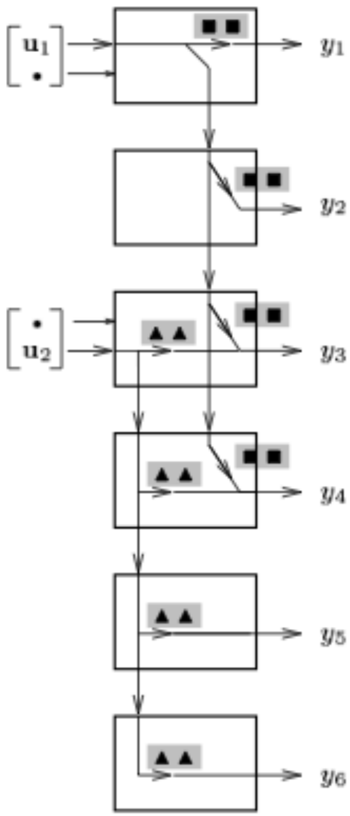
code structure Ts, with s the signal:



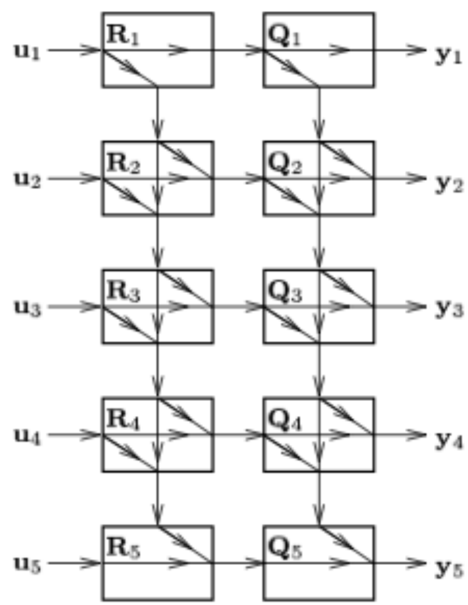
structure of the code matrix and efficient least-squares inversion

$\mathbf{T} =$

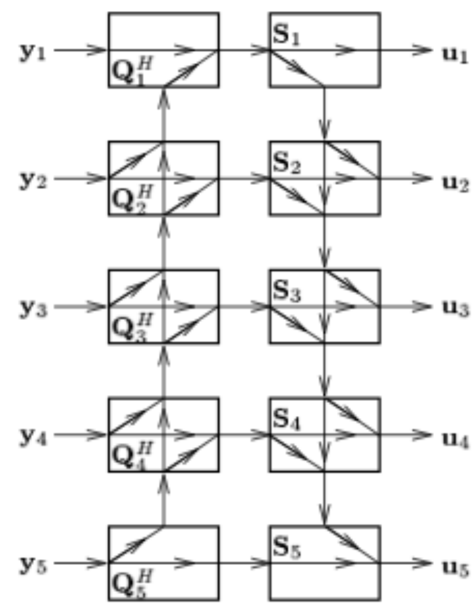




$$\mathbf{T} = \tilde{\mathbf{Q}}_N \cdots \tilde{\mathbf{Q}}_1 \tilde{\mathbf{R}}_N \cdots \tilde{\mathbf{R}}_1$$



$$\mathbf{T}^{\dagger} = \tilde{\mathbf{S}}_N \cdots \tilde{\mathbf{S}}_1 \tilde{\mathbf{Q}}_1^H \cdots \tilde{\mathbf{Q}}_N^H$$



much better than the traditional matched filter, both in accuracy and in complexity!

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Envoy

it's a two way street: matrices for systems system theory for matrices

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